Free Heyting algebras via duality

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Free Boolean algebras

Boolean algebras form a variety (equational class) BA which provides the algebraic semantics of the classical propositional calculus (CPC).

Terms in the language of Boolean algebras correspond to formulas in the language of CPC.

Theorem (Algebraic completeness of CPC)

Let t be a term corresponding to a formula φ. Then

$$\mathsf{BA} \models t = 1 \qquad \textit{iff} \qquad \mathsf{CPC} \vdash \varphi.$$

The right to left implication is a straightforward verification. What about the other implication?

Let X be a set of propositional variables and consider the set $\mathsf{Form}(X)$ of formulas over the variables in X.

Define $\sim$ on $\mathsf{Form}(X)$ by

$$\varphi \sim \psi \quad \text{iff} \quad \mathsf{CPC} \vdash \varphi \leftrightarrow \psi.$$

The Lindenbaum-Tarski algebra for CPC over X is $\mathsf{Form}(X)/{\sim}$.

$$[\varphi] \wedge [\psi] = [\varphi \wedge \psi] \qquad [\varphi] \vee [\psi] = [\varphi \vee \psi] \qquad \neg[\varphi] = [\neg\varphi]$$

$$1 = [\top] \qquad 0 = [\bot]$$

Proposition

$\mathsf{Form}(X)/{\sim}$ is a Boolean algebra.

Note that $[\varphi] \leq [\psi]$ iff $\mathsf{CPC} \vdash \varphi \to \psi$.

The algebraic completeness of CPC is then a consequence of the following lemma.

Lemma

If $t(x_1, \ldots, x_n)$ is a term corresponding to a formula $\varphi(p_1, \ldots, p_n)$ over X, then

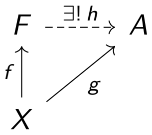
$$\mathrm{Form}(X)/{\sim} \vDash t = 1 \qquad \textit{implies} \qquad \mathrm{CPC} \vdash \varphi.$$

Proof (sketch): We have $t([p_1], \ldots, [p_n]) = [\varphi(p_1, \ldots, p_n)]$, and hence $[\varphi(p_1, \ldots, p_n)] = 1$. Therefore,

$$\mathrm{CPC} \vdash \varphi \leftrightarrow \top,$$

which yields $\mathrm{CPC} \vdash \varphi$.

Let V be a variety and X a set. An algebra $F \in \mathsf{V}$ is said to be free over X if there exists a function $f\colon X \to F$ such that for any $A \in V$ and function $g\colon X \to A$, there is a unique homomorphism $h\colon F \to A$ with $g = h \circ f$.



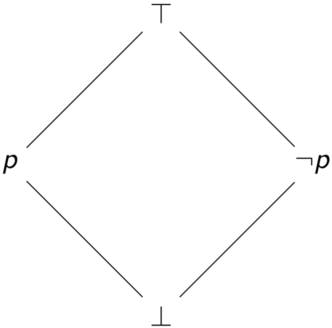
Free algebras over a given set are unique up to isomorphism, so we will talk about **the** free V-algebra over X and denote it by $\mathsf{F}_{\mathsf{V}}(X)$.

If $|X| = |Y|$, then $\mathsf{F}_{\mathsf{V}}(X) \cong \mathsf{F}_{\mathsf{V}}(Y)$. So, we will often write $\mathsf{F}_{\mathsf{V}}(|X|)$ instead of $\mathsf{F}_{\mathsf{V}}(X)$.

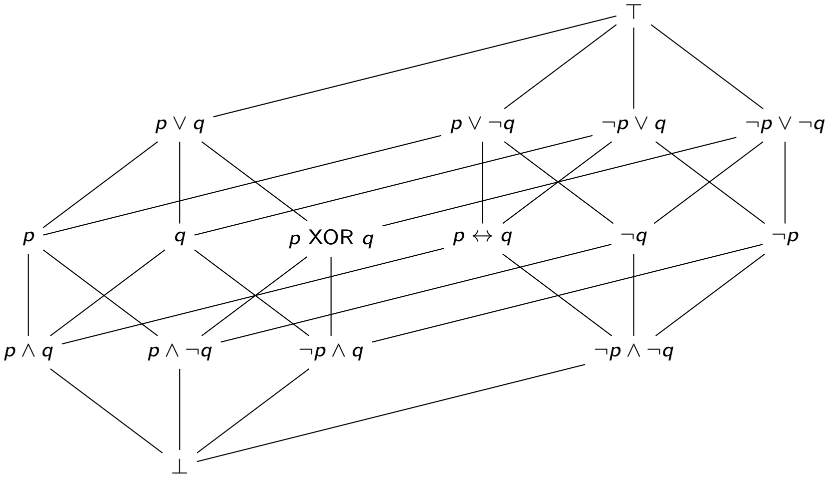
Theorem

The Lindenbaum-Tarski algebra $\mathrm{Form}(X)/{\sim}$ *is the Boolean algebra free over* X.

$F_{BA}(1)$



$F_{BA}(2)$



What about $\mathsf{F}_{\mathsf{BA}}(3),\ \mathsf{F}_{\mathsf{BA}}(4), \ldots$?

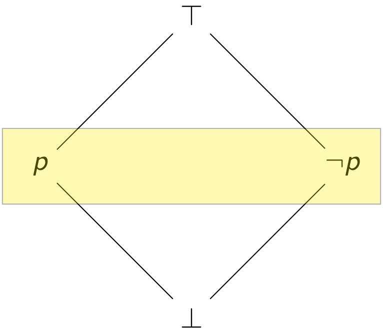
When $n \in \omega$, then $\mathsf{F}_{\mathsf{BA}}(n)$ is finite; i.e., BA is a locally finite variety.

Indeed, every formula is equivalent to one in disjunctive normal form: a disjunction of formulas of the form

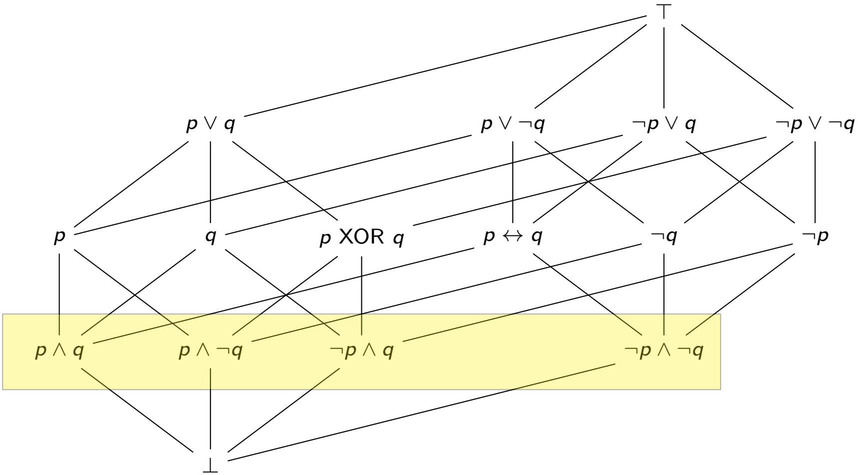
$$(\neg)p_1 \wedge (\neg)p_2 \wedge \cdots \wedge (\neg)p_n.$$

Recall that an atom of a Boolean algebra B is a minimal nonzero element of B.

Thus, $\mathsf{F}_{\mathsf{BA}}(n)$ has exactly 2^n atoms, which are the equivalence classes of the formulas of the form above.



The atoms of $\mathsf{F}_{\mathsf{BA}}(1)$ correspond to the possible situations that can be described with one proposition p: either p holds or it doesn't.

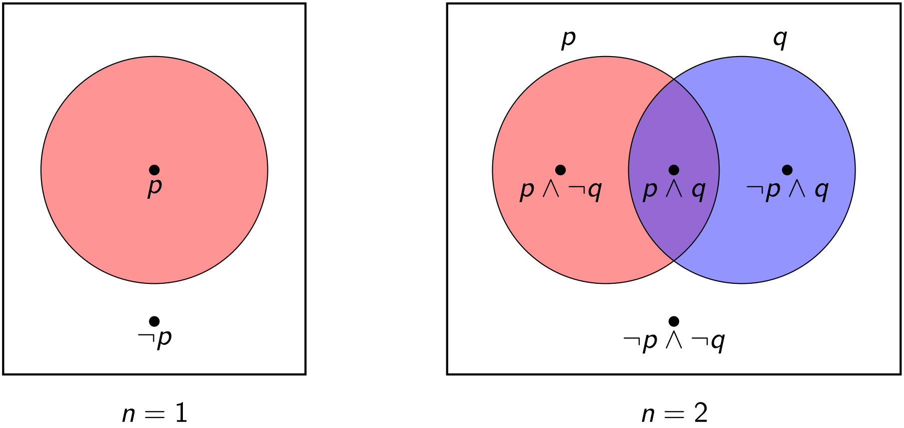


The atoms of $F_{BA}(2)$ correspond to the possible situations that can be described with 2 independent propositions p and q.

Every finite Boolean algebra is isomorphic to the powerset of the set of its atoms.

Theorem

$\mathsf{F}_{\mathsf{BA}}(n)$ *is isomorphic to* $\mathcal{P}(2^n)$, *and so it has* 2^{2^n} *elements.*



Free Boolean algebras via Stone duality

What about $\mathsf{F}_{\mathsf{BA}}(\kappa)$ for an infinite cardinal κ? Then $\mathsf{F}_{\mathsf{BA}}(\kappa)$ is infinite: it has cardinality κ.

Proposition

$\mathsf{F}_{\mathsf{BA}}(\kappa)$ doesn't have atoms for every infinite κ.

Proof (sketch): suppose that $[\varphi] \in \mathsf{F}_{\mathsf{BA}}(\kappa)$ is an atom. Consider a propositional letter p that doesn't appear in φ. Then $0 \neq [\varphi \wedge p] < [\varphi]$.

We will replace atoms with ultrafilters.

A filter F of a Boolean algebra B is a subset $F \subseteq B$ such that:

- $1 \in F$,
- $a \in F$ and $a \leq b$ imply $b \in F$,
- $a, b \in F$ implies $a \wedge b \in F$.

A proper filter F is called an ultrafilter if the following equivalent conditions hold:

- F is a maximal proper filter of B,
- $a \in F$ or $\neg a \in F$ for every $a \in B$,
- $a \vee b \in F$ implies $a \in F$ or $b \in F$.

Atoms are not suitable to deal with infinite Boolean algebras. We'll work with ultrafilters instead.

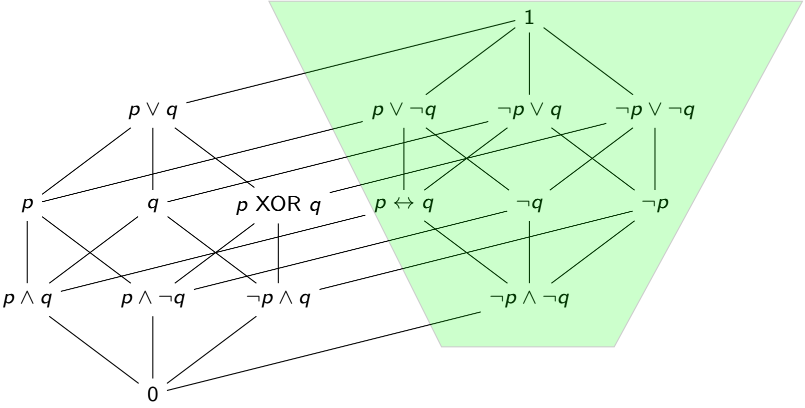
- If $a \in B$ is an atom, then $\uparrow a := \{b : a \leq b\}$ is an ultrafilter. Every principal ultrafilter is of that form.
- If B is finite, then every ultrafilter of B is principal.
- When B is infinite, there exist nonprincipal ultrafilters.

A theory T (i.e., set of propositional formulas) is deductively closed if $T \vdash_{\mathsf{CPC}} \varphi$, then $\varphi \in T$.

T is called complete if it is consistent and $T \vdash_{\mathsf{CPC}} \varphi$ or $T \vdash_{\mathsf{CPC}} \neg\varphi$ for every φ.

Filters of $\mathsf{F}_{\mathsf{BA}}(\kappa)$ correspond to deductively closed theories in κ variables.

Ultrafilters of $\mathsf{F}_{\mathsf{BA}}(\kappa)$ correspond to complete deductively closed theories.



Ultrafilters of a free Boolean algebra can then be thought of as possible worlds: a world is completely described by the collection of formulas that are true in that world.

The larger κ is, the more propositions you have to differentiate between possible worlds.

This is a bridge between syntax and semantics that will lead us to Stone duality.

Can we get a similar intuition about ultrafilters of an arbitrary Boolean algebra?

Let B be a Boolean algebra. Then there exists an onto homomorphism $\mathsf{F}_{\mathsf{BA}}(\kappa) \to B$ for some cardinal κ.

It follows that $B \cong \mathsf{F}_{\mathsf{BA}}(\kappa)/F$ for a filter F of $\mathsf{F}_{\mathsf{BA}}(\kappa)$.

Let T be the theory corresponding to F. Then B can be thought of as a Lindenbaum-Tarski algebra modulo T. More precisely, B is isomorphic to the Boolean algebra $\mathsf{Form}(\kappa)/\!\sim$, where

$$\varphi \sim \psi \qquad \text{iff} \qquad T \vdash_{\mathsf{CPC}} \varphi \leftrightarrow \psi$$

and the operations are defined as in the Lindenbaum-Tarski algebras.

Since $B \cong \mathsf{F}_{\mathsf{BA}}(\kappa)/F$, ultrafilters of B correspond to ultrafilters of $\mathsf{F}_{\mathsf{BA}}(\kappa)$ containing F.

So, ultrafilters of B can be thought of as the complete theories extending T. Thus, they correspond to the possible worlds in which T is true; i.e., the models of T.

The set $\mathrm{Ult}(B)$ of ultrafilters of B is naturally equipped with a topology.

Intuition: a world x is "close" to a set of worlds S if every formula true in x is also true in some world in S.

Let B be a Boolean algebra. We equip the set $\mathrm{Ult}(B)$ of ultrafilters of B with the topology generated by the basis $\{\sigma(a) : a \in B\}$, where

$$\sigma(a) := \{F \in \mathrm{Ult}(B) : a \in F\}.$$

Syntax $\to$ semantics: points of $\mathrm{Ult}(B)$ are possible worlds and the elements of B are formulas. Then $\sigma(a)$ is the set of worlds in which the formula a holds.

Then $\text{Ult}(B)$ is a topological space that is:

- Hausdorff,
- compact,
- zero-dimensional: the clopen (closed and open) subsets form a basis.

The topological spaces satisfying these three properties are called Stone (or Boolean) spaces.

Let X be a Stone space and $\text{Clop}(X)$ the collection of its clopen subsets. Then $(\text{Clop}(X), \cap, \cup, -, \varnothing, X)$ is a Boolean algebra.

Semantics $\rightarrow$ syntax: points of X are possible worlds and the elements of $\text{Clop}(X)$ are formulas. Each clopen V is the collection of worlds in which it is "true".

- Let BA be the category of Boolean algebras and Boolean homomorphisms.
- Let Stone be the category of Stone spaces and continuous functions.

We obtain contravariant functors $\mathsf{Ult}\colon \mathsf{BA} \to \mathsf{Stone}$ and $\mathsf{Clop}\colon \mathsf{Stone} \to \mathsf{BA}$ by setting:

- If $h\colon A \to B$, then $\mathsf{Ult}(h) \coloneqq h^{-1}\colon \mathsf{Ult}(B) \to \mathsf{Ult}(A)$.
- If $f\colon X \to Y$, then $\mathsf{Clop}(f) \coloneqq f^{-1}\colon \mathsf{Clop}(Y) \to \mathsf{Clop}(X)$.

Theorem (Stone duality)

Ult *and* Clop *establish a dual equivalence between* BA *and* Stone*.*

In particular,

- $B \cong \mathsf{Clop}(\mathsf{Ult}(B))$ for every $B \in \mathsf{BA}$,
- $X \cong \mathsf{Ult}(\mathsf{Clop}(X))$ for every $X \in \mathsf{Stone}$,

and these isomorphisms are natural, meaning that they behave well with the morphisms.

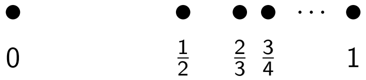
Ultrafilters of a Boolean algebra B correspond to

- Boolean homomorphisms onto the 2-element Boolean algebra.
  A possible world correspond to a map that tells whether a formula is true or false in that world.
- maximal ideals.
  A possible world correspond to the collection of formulas that are false in that world.
- atoms (or coatoms) when the Boolean algebra is finite.

Some examples

- Finite sets with the discrete topology are dual to finite Boolean algebras. In this case, $\mathrm{Clop}(X) = \mathcal{P}(X)$.
- The Stone-Čech compactification of a discrete space X is dual to $\mathcal{P}(X)$.
- The one point compactification of any infinite discrete space X is dual to the Boolean subalgebra of $\mathcal{P}(X)$ consisting of the finite and cofinite subsets of X.

  E.g., when X is countable the one point compactification of X is homeomorphic to the subspace $\{1 - 1/n : n \geq 1\} \cup \{1\}$ of $\mathbb{R}$.



Boolean algebras	Stone spaces
1-1 homomorphisms	Onto continuous maps
Subalgebras	Stone equivalence relations
Onto homomorphisms	1-1 continuous maps
Homomorphic images	Closed subsets
Binary products	Binary disjoint unions
Coproducts	Cartesian products
Atoms	Isolated points

Let $\mathbf{2}$ be the 2-element discrete topological space. We have seen that $\mathsf{F}_{\mathsf{BA}}(n)$ is dual to $\mathbf{2}^n$. In particular, $\mathsf{F}_{\mathsf{BA}}(1)$ is dual to $\mathbf{2}$.

Theorem

Let κ be a cardinal. Then $\mathsf{F}_{\mathsf{BA}}(\kappa)$ is dual to $\mathbf{2}^\kappa$.

Proof. $\mathsf{F}_{\mathsf{BA}}(\kappa)$ is the coproduct of κ copies of $\mathsf{F}_{\mathsf{BA}}(1)$. Therefore, $\mathsf{F}_{\mathsf{BA}}(\kappa)$ is dual to the cartesian product of κ copies of $\mathbf{2}$; i.e., $\mathbf{2}^\kappa$. □

Note that $\mathbf{2}^\omega$ is homeomorphic to the Cantor space.

$\mathbf{2}^\kappa$ is sometimes called a generalized Cantor space and can be thought of as the vertexes of a κ-dimensional hypercube.

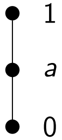
Free distributive lattices via Priestley duality

A filter F of a (bounded) distributive lattice D is called prime if

$$a \vee b \in F \quad \text{implies} \quad a \in F \text{ or } b \in F.$$

Let $\mathrm{Spec}(D)$ be the spectrum of D; i.e., the collection of prime filters of D.

Note that prime filters do not coincide with maximal proper filters in a distributive lattice. E.g., the 3-element chain has two prime filters, but only one of them is maximal.



When D is finite, prime filters correspond to join-irreducibles, while maximal proper filters correspond to atoms.

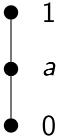
Spec(D) can be equipped with different topologies. We define a topology that makes Spec(D) into a Stone space.

Equip Spec(D) with the topology generated by the subbasis

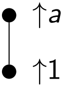
$$\{\sigma(a) : a \in D\} \cup \{\mathrm{Spec}(D) \setminus \sigma(a) : a \in D\},$$

where $\sigma(a) = \{F \in \mathrm{Spec}(D) : a \in F\}$.

We also order Spec(D) with the inclusion order $\subseteq$. This makes Spec(D) into an topological space equipped with a partial order; i.e., an ordered topological space.



D



Spec(D)

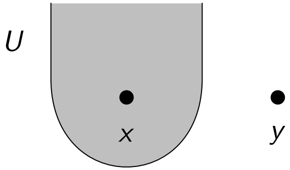
A subset U of a partially ordered set X is called an upset if it is upward closed: $x \in U$ and $x \leq y$ imply $y \in U$. Similarly, downward closed subsets are called downsets.

Recall that a subset of a topological space is called clopen if it is both closed and open. Let $\mathrm{ClopUp}(X)$ be the collection of clopen upsets of a partially ordered topological space X, which is a distributive lattice once ordered by inclusion.

A Priestley space is a partially ordered topological space X such that

- X is compact;
- it satisfies the Priestley separation axiom:

$$x \not\leq y \quad \text{implies} \quad \exists U \in \mathrm{ClopUp}(X) \text{ s.t. } x \in U \text{ and } y \notin U.$$



- Let DL be the category of distributive lattices and (bounded) lattice homomorphisms.
- Let Pries be the category of Priestley spaces and order preserving continuous functions.

We obtain contravariant functors $\mathsf{ClopUp}\colon \mathsf{Pries} \to \mathsf{DL}$ and $\mathsf{Spec}\colon \mathsf{DL} \to \mathsf{Pries}$ by setting

- If $h\colon A \to B$, then $\mathsf{Spec}(h) := h^{-1}\colon \mathsf{Spec}(B) \to \mathsf{Spec}(A)$.
- If $f\colon X \to Y$, then $\mathsf{ClopUp}(f) := f^{-1}\colon \mathsf{ClopUp}(Y) \to \mathsf{ClopUp}(X)$.

Theorem (Priestley duality (Priestley 1970, 1972))

Spec *and* ClopUp *establish a dual equivalence between* DL *and* Pries.

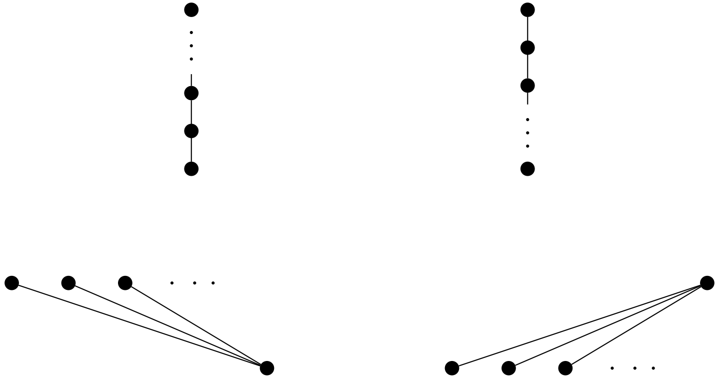
In particular,

- $D \cong \mathsf{ClopUp}(\mathsf{Spec}(D))$ for every $D \in \mathsf{DL}$,
- $X \cong \mathsf{Spec}(\mathsf{ClopUp}(X))$ for every $X \in \mathsf{Pries}$,

and these isomorphisms are natural, meaning that they behave well with the morphisms.

Some examples

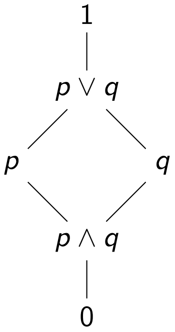
- Finite posets with the discrete topology are duals to finite distributive lattices. In this case, $\mathrm{ClopUp}(X)$ is the set of upsets of X.
- Every Stone space is a Priestley space with the identity order. So, Priestley duality can be thought of as an extension of Stone duality
- The same Stone space can have many orders making it into a Priestley space.



Distributive lattices	Priestley spaces
1-1 homomorphisms	Onto continuous order preserving maps
Subalgebras	Priestley quasiorders
Onto homomorphisms	1-1 continuous order preserving and reflecting maps
Homomorphic images	Closed subsets
Binary products	Binary disjoint unions
Coproducts	Cartesian products with product order and product topology

$F_{DL}(\kappa)$ is obtained as the set of terms in κ variables in the language of bounded lattices up to equivalence in DL.

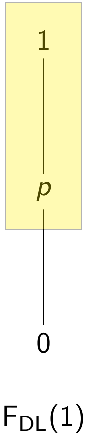
$F_{DL}(1)$

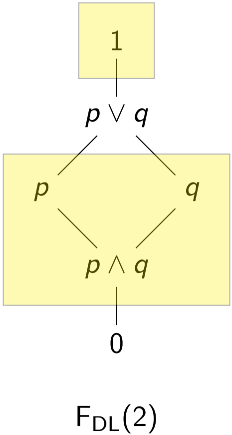


$F_{DL}(2)$

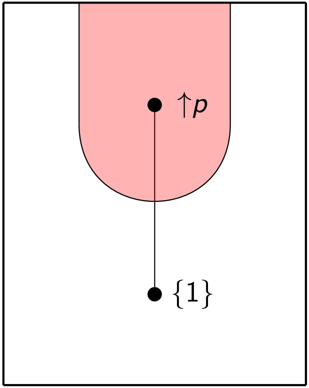
DL is also locally finite: $F_{DL}(n)$ is finite for every $n \in \omega$.

The prime filters of $\mathsf{F}_{\mathsf{DL}}(n)$ are the upsets of its join irreducible elements.

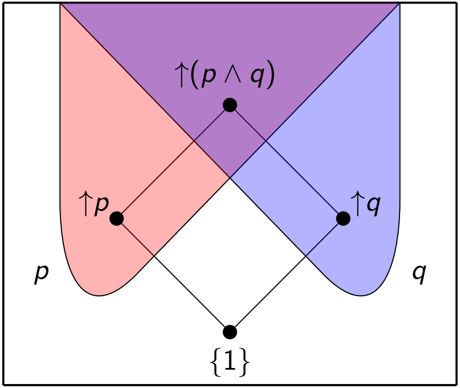




We can reconstruct $F_{DL}(n)$ from its spectrum.



$n = 1$



$n = 2$

Let $\mathbb{2}$ be the 2-element chain with the discrete topology. We have seen that $\mathsf{F}_{\mathsf{DL}}(1)$ is dual to $\mathbb{2}$.

Theorem

Let κ be a cardinal. Then $\mathsf{F}_{\mathsf{DL}}(\kappa)$ is dual to $\mathbb{2}^\kappa$.

Proof. $\mathsf{F}_{\mathsf{DL}}(\kappa)$ is the coproduct of κ copies of $\mathsf{F}_{\mathsf{DL}}(1)$. Therefore, $\mathsf{F}_{\mathsf{DL}}(\kappa)$ is dual to the product of κ copies of $\mathbb{2}$; i.e., $\mathbb{2}^\kappa$. □

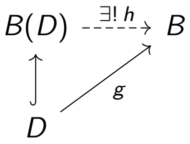
$\mathbb{2}^\kappa$ can be thought of as the vertexes of a κ-dimensional hypercube ordered componentwise.

Open problem: What is the cardinality of $\mathsf{F}_{\mathsf{DL}}(n)$ for $n \geq 10$? Equivalently, how many upsets of $\mathbb{2}^n$ are there? (look up Dedekind numbers)

We can forget the order on a Priestley space and obtain a Stone space. This yields a "forgetful" functor Pries $\to$ Stone.

On the algebraic side, this corresponds to a functor DL $\to$ BA that is left adjoint to the inclusion functor BA $\hookrightarrow$ DL.

This functor sends a distributive lattice D to its free Boolean extension $B(D)$ (i.e., the Boolean algebra freely generated over D) having the following universal property:



for every lattice homomorphism $g \colon D \to B$ into a Boolean algebra there is a unique Boolean homomorphism $h \colon B(D) \to B$ making the diagram commute.

The free Boolean extension of $\mathsf{F}_{\mathsf{DL}}(\kappa)$ is $\mathsf{F}_{\mathsf{BA}}(\kappa)$.

This is reflected in the fact that forgetting the order of $\mathbb{2}^{\kappa}$ gives exactly $\mathbf{2}^{\kappa}$.

Heyting algebras and Esakia duality

Intuitionistic propositional logic

Intuitionistic logic is the logic of constructive mathematics and has its origins in Brouwer's criticism of the use of the principle of the excluded middle.

It is obtained by weakening the principles of classical logic via the rejection of the law of excluded middle ($p \vee \neg p$).

We denote by IPC the intuitionistic propositional calculus and the connectives of IPC by $\wedge, \vee, \rightarrow, \bot, \top$. The negation $\neg$ is defined as an abbreviation $\neg\varphi \coloneqq \varphi \rightarrow \bot$.

BHK interpretation

In classical logic the propositional connectives have a truth functional interpretation:

- $\varphi \wedge \psi$ is true iff φ is true and ψ is true,
- $\varphi \vee \psi$ is true iff φ is true or ψ is true,
- $\neg\varphi$ is true iff φ is not true

The BHK-interpretation of intuitionistic logic is based on the notion of (informal) constructive proof instead of truth:

- A proof of $\varphi \wedge \psi$ consists of a proof of φ and a proof of ψ,
- A proof of $\varphi \vee \psi$ consists of a proof of φ or a proof of ψ,
- A proof of $\varphi \to \psi$ consists of a method of converting any proof of φ into a proof of ψ,
- No proof of $\bot$ exists,
- A proof of $\neg\varphi$ then consists of a method of converting any proof of φ into a proof of a contradiction.

Examples of invalid formulas in IPC are

- $\varphi \vee \neg\varphi$ (law of the excluded middle)
- $\neg\neg\varphi \to \varphi$ (double negation law)
- $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$ (contraposition)
- $\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$ (one of the De Morgan laws)
- $\neg\varphi \vee \neg\neg\varphi$ (law of the weak excluded middle)
- $(\varphi \to \psi) \to (\neg\varphi \vee \psi)$

The following classical tautologies are also valid in IPC:

- $\varphi \to \neg\neg\varphi$
- $\neg\varphi \leftrightarrow \neg\neg\neg\varphi$
- $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
- $(\neg\varphi \vee \psi) \to (\varphi \to \psi)$
- the distributivity laws for $\wedge$ and $\vee$
- the remaining three implications in the De Morgan laws

A Heyting algebra $(H, \wedge, \vee, \to, 0, 1)$ is a distributive lattice equipped with a binary operation $\to$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c$$

for any $a, b, c \in H$.

Let HA be the variety of Heyting algebras.

Examples:

- Boolean algebras are Heyting algebras with $a \to b = \neg a \vee b$.
- Let X be a topological space. Then the collection $\mathcal{O}(X)$ of open subsets of X ordered by inclusion is a Heyting algebra, where $U \to V = \text{int}((X \setminus U) \cup V)$.
- Let X be a poset. Then the collection $\text{Up}(X)$ of upsets of X ordered by inclusion is a Heyting algebra, where $U \to V = X \setminus {\downarrow}(U \setminus V)$ with ${\downarrow}Y = \{x \in X : x \leq y \text{ for some } y \in Y\}$.

Similarly to what we did for Boolean algebras, we can define Lindenbaum-Tarski algebras for IPC as $\mathsf{Form}(X)/\sim$, where

$$\varphi \sim \psi \quad \text{iff} \quad \mathsf{IPC} \vdash \varphi \leftrightarrow \psi.$$

Theorem (Algebraic completeness of IPC)

Let t be a term corresponding to a formula φ. Then

$$\mathsf{HA} \vDash t = 1 \qquad \textit{iff} \qquad \mathsf{IPC} \vdash \varphi.$$

Lindenbaum-Tarski algebras for IPC are exactly the free Heyting algebras.

Recap

A Stone space is a topological space that is

- Hausdorff,
- compact,
- zero-dimensional: the clopen (closed and open) subsets form a basis.

Theorem (Stone duality)

Ult *and* Clop *establish a dual equivalence between* BA *and* Stone*.*

Let $\mathbf{2}$ be the 2-element discrete topological space.

Theorem

$\mathsf{F}_{\mathsf{BA}}(\kappa)$ *is dual to the Stone space* $\mathbf{2}^{\kappa}$*.*

Recap

A Priestley space is a partially ordered topological space X such that

- X is compact;
- it satisfies the Priestley separation axiom:

$$x \not\leq y \quad \text{implies} \quad \exists U \in \mathrm{ClopUp}(X) \text{ s.t. } x \in U \text{ and } y \notin U.$$

Theorem (Priestley duality)

Spec *and* ClopUp *establish a dual equivalence between* DL *and* Pries.

Let 2 be the 2-element chain with the discrete topology.

Theorem

$\mathsf{F}_{\mathsf{DL}}(\kappa)$ *is dual to the Priestley space* 2^κ.

Recap

A Heyting algebra $(H, \wedge, \vee, \rightarrow, 0, 1)$ is a distributive lattice equipped with a binary operation $\rightarrow$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \rightarrow c$$

for any $a, b, c \in H$.

Theorem (Algebraic completeness of IPC)

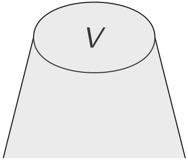
Let t be a term corresponding to a formula φ. Then

$$\mathsf{HA} \vDash t = 1 \qquad \textit{iff} \qquad \mathsf{IPC} \vdash \varphi.$$

Priestley duality restricts to Esakia duality for Heyting algebras.

An Esakia space is a Priestley space X that satisfies the additional condition:

- If V is clopen (open), then $\downarrow V := \{x \in X : x \leq y \text{ for some } y \in V\}$ is clopen (open).



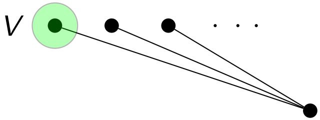
They can equivalently be defined as Stone spaces X equipped with a partial order such that:

- $\uparrow x := \{y \in X : x \leq y\}$ is closed for every $x \in X$,
- $\downarrow V$ is clopen for every clopen subset V of X.

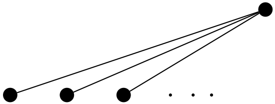
Which of the following are Esakia spaces?

Esakia

Esakia



not Esakia



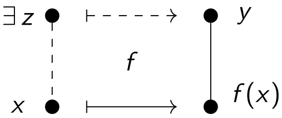
Esakia

V is open but $\downarrow V$ is not open.

Not every lattice homomorphism between Heyting algebras is a Heyting homomorphism because it doesn't have to preserve implications.

A map $f\colon X \to Y$ between posets is called a p-morphism if it satisfies one of the following equivalent conditions:

- $f[\uparrow x] = \uparrow f(x)$ for every $x \in X$,
- $f^{-1}[\downarrow y] = \downarrow f^{-1}(y)$ for every $y \in Y$,
- f is order preserving and for all $x \in X$ and $y \in Y$ if $f(x) \leq y$, then there is $z \geq x$ such that $f(z) = y$.



Let Esa be the subcategory of Pries consisting of Esakia spaces and continuous p-morphisms.

If X is an Esakia space, then $\mathrm{ClopUp}(X)$ is a Heyting algebra with implication defined as

$$U \to V = X \setminus {\downarrow}(U \setminus V).$$

Then $U \to V$ is the collection of points $x \in X$ such that

for every $y \geq x$ (if $y \in U$ then $y \in V$).

Moreover, $U \to V$ is the largest upset contained in $(X \setminus U) \cup V$.

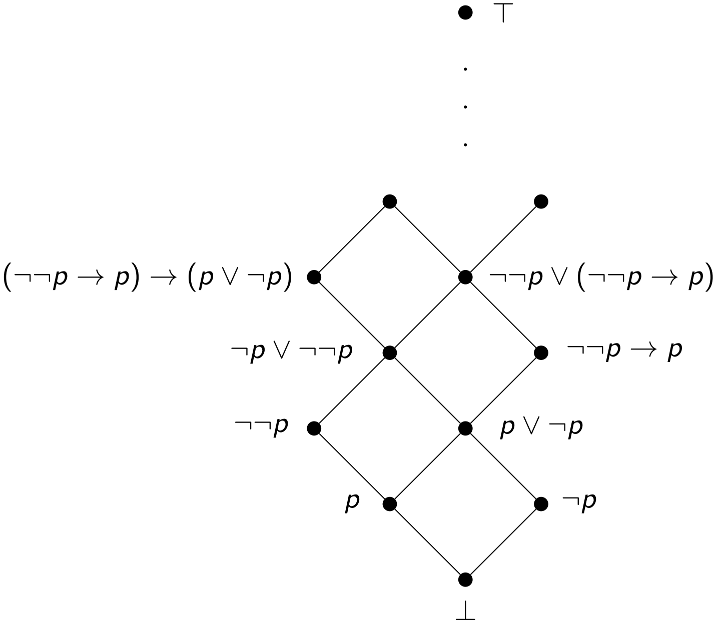
Theorem (Esakia duality (Esakia 1974))

The contravariant functors ClopUp *and* Spec *restrict to a dual equivalence between* Esa *and* HA.

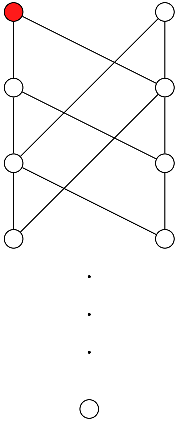
Heyting algebras	Esakia spaces
1-1 homomorphisms	Onto continuous p-morphisms
Subalgebras	Bisimulation equivalences
Onto homomorphisms	1-1 continuous p-morphisms
Homomorphic images	Closed upsets
Binary products	Binary disjoint unions
Coproducts	(it's complicated)

What does $F_{HA}(\kappa)$ look like? What about its Esakia dual?

$F_{HA}(1)$ is also known as the Rieger-Nishimura lattice.



The Esakia dual of $F_{HA}(1)$ looks like this. The red point represents the clopen upset generating $F_{HA}(1)$.



All its points are isolated except for the bottom.
This poset (minus the bottom) is known as the Rieger-Nishimura ladder.

What about $\mathsf{F}_{\mathsf{HA}}(\kappa)$ for $\kappa > 1$?

Already $\mathsf{F}_{\mathsf{HA}}(2)$ is extremely complicated.

Theorem

The Esakia dual of $\mathsf{F}_{\mathsf{HA}}(2)$ *has the cardinality of the continuum.*

However, we will see that when $n \in \omega$, we can understand a portion of the Esakia dual of $\mathsf{F}_{\mathsf{HA}}(n)$ that contains enough information to reconstruct the whole $\mathsf{F}_{\mathsf{HA}}(n)$.

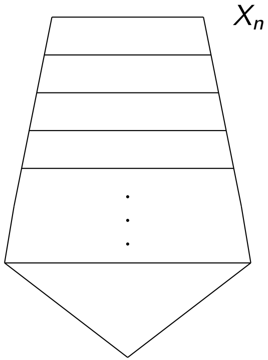
A point x of a poset has depth n if the greatest size of chains in $\uparrow x$ is n. We say that x is of finite depth if it is of depth n for some $n \in \omega$.

Theorem

Let X_n be the Esakia dual of $\mathsf{F_{HA}}(n)$ for $n \in \omega$.

- *X_n contains finitely many points of depth m for every $m \in \omega$.*
- *For all m, each point of X_n that is not of finite depth is below a point of depth m.*
- *A point $x \in X_n$ is of finite depth iff $\uparrow x$ is finite iff x is isolated.*
- *Points of finite depth form an open upset that is topologically dense.*
- *X_n has a least element.*

The points of finite depth constitute the "upper part" of X_n, which is an open upset dense in X_n. Moreover, the points in the upper part are exactly the isolated points of X_n.



The points that are not of finite depth constitute the lower part, usually called the remainder. For $n \geq 2$ the remainder is extremely complicated to understand. However, it always has a least element.

Our goal is to describe the upper part of X_n. Note that the topology doesn't play any role there because all its points are isolated.

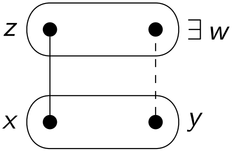
We will see that this portion of X_n can be constructed recursively layer-by-layer and is called the n-universal model for IPC.

The idea behind this construction is to "glue together" all the duals of finite n-generated Heyting algebras in a "universal way".

We will use the coloring theorem to dually describe finite sets of generators of Heyting algebras.

An equivalence relation E on a Stone space X is said to be a Stone equivalence relation if X/E is a Stone space.

A Stone equivalence relation E on an Esakia space X is a bisimulation equivalence (a.k.a. correct partition) when for every $x, y, z \in X$ if xEy and $x \leq z$, then there is $w \in X$ such that $y \leq w$ and zEw.



Proposition

Let H be a Heyting algebra and X its Esakia dual. Then subalgebras of H correspond to bisimulation equivalences on X.

Given a bisimulation equivalence E on X, the corresponding subalgebra of $\mathrm{ClopUp}(X)$ consists the E-saturated clopen upsets of X; i.e., the $U \in \mathrm{ClopUp}(X)$ such that $E[U] = U$.

Let X be a poset. We call n-coloring a map $c\colon X \to \mathcal{P}(n)$ such that $x \leq y$ implies $c(x) \subseteq c(y)$.

If X is an Esakia space and $U_1, \ldots, U_n \in \mathrm{ClopUp}(X)$, then we can define a coloring by $c(x) = \{i : x \in U_i\}$. We think of $c(x)$ as the color of x.

There is a correspondence between continuous (wrt the discrete topology on $\mathcal{P}(n)$) n-colorings and tuples $U_1, \ldots, U_n \in \mathrm{ClopUp}(X)$.

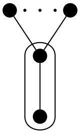
Theorem (Coloring theorem (Esakia-Grigolia 1977))

$U_1, \ldots, U_n$ generate $\mathrm{ClopUp}(X)$ iff every nonidentity bisimulation equivalence on X identifies two points of different colors.

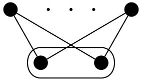
Proof (sketch): A bisimulation equivalence only identifies points of the same color iff it corresponds to a subalgebra of $\mathrm{ClopUp}(X)$ containing $U_1, \ldots, U_n$. So, the condition on the right is equivalent to saying that no proper subalgebra of $\mathrm{ClopUp}(X)$ contains $U_1, \ldots, U_n$; i.e., $U_1, \ldots, U_n$ generate $\mathrm{ClopUp}(X)$. □

Let X be a **finite** poset (Esakia space). Then every nonidentity bisimulation equivalence contains one of the form:

Alpha reduction: E is the identity on X except for a pair of elements $x, y \in X$ such that xEy and $\uparrow x \setminus \{x\} = \uparrow y$.



Beta reduction: E is the identity on X except for a pair of distinct elements $x, y \in X$ such that xEy and $\uparrow x \setminus \{x\} = \uparrow y \setminus \{y\}$.



Theorem (Coloring theorem for finite posets)

Let X be a finite poset. Then $U_1, \ldots, U_n$ generate $\mathrm{Up}(X)$ iff every alpha or beta reduction on X identifies two points of different colors.

When the n-coloring of a finite poset satisfies the condition of the coloring theorem, the colored poset is called irreducible.

Finite n-colored irreducible posets correspond to n-generated finite Heyting algebras.

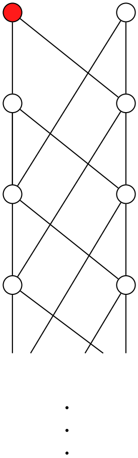
Up to isomorphism, there is a unique n-colored poset U_n with the following property: every finite n-colored irreducible poset is isomorphic (as a colored poset) to a unique upset of U_n.

U_n is called the n-universal model of IPC. The study of U_n originates with the works of Shehtman (1978) and Bellissima (1986).

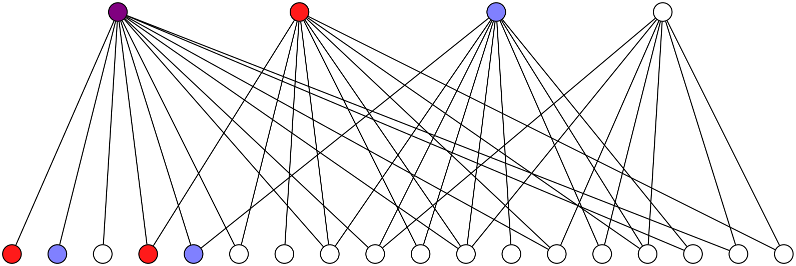
The *n-universal model* U_n is built layer-by-layer starting from the top layer by adding all the possible points satisfying the following conditions:

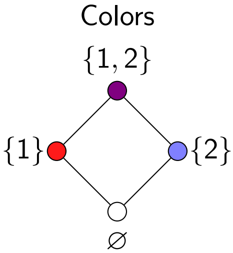
- the top layer contains 2^n points, one for each color;
- if a point has only one immediate successor, then its color should be strictly smaller than the one of its successor (alpha reduction);
- two points of the same color cannot have the same elements as immediate successors (beta reduction).

The 1-universal model U_1 is the Rieger-Nishimura ladder.



The layers of the 2-universal model U_2 increase very quickly.





How big is its third layer? More than 250 000 points!

U_n is the upper part of the Esakia dual X_n of $\mathsf{F}_{\mathsf{HA}}(n)$.

Theorem

- *U_n is isomorphic to the upset of X_n consisting of its isolated points.*
- *The coloring of U_n is the restriction of the coloring of X_n induced by the free generators of $\mathsf{F}_{\mathsf{HA}}(n) \cong \mathsf{ClopUp}(X_n)$.*

Proof (idea): The finite upsets of X_n are dual to the finite homomorphic images of $\mathsf{F}_{\mathsf{HA}}(n)$, which are the n-generated finite Heyting algebras. Moreover, a homomorphism from $\mathsf{F}_{\mathsf{HA}}(n)$ onto a finite Heyting algebra is completely determined by the image of the free generators of $\mathsf{F}_{\mathsf{HA}}(n)$.

The "palette" of colors we used to build U_n looks like $\mathbb{2}^n$. Why?

This is because $\mathsf{F}_{\mathsf{HA}}(n)$ can be seen as the free Heyting algebra over the distributive lattice $\mathsf{F}_{\mathsf{DL}}(n)$ and the coloring $c\colon X_n \to \mathbb{2}^n$ induced by the free generators is the Priestley dual of the universal lattice homomorphism $\mathsf{F}_{\mathsf{DL}}(n) \to \mathsf{F}_{\mathsf{HA}}(n)$.

One can replace the "palette" $\mathbb{2}^n$ with other finite posets to study Heyting algebras free over arbitrary finite distributive lattices.

We also have that $\max X_n$ looks like $\mathbf{2}^n$. Why?

This is because if $X \in \mathsf{Esa}$ is dual to $H \in \mathsf{HA}$, then $\max X$ is a closed upset corresponding to the regularization $R(H)$ of H, which is a Boolean algebra.

Mapping H to $R(H)$ defines a functor $\mathsf{HA} \to \mathsf{BA}$ that is left adjoint to the inclusion $\mathsf{BA} \to \mathsf{HA}$. This implies that the regularization of $\mathsf{F}_{\mathsf{HA}}(\kappa)$ is isomorphic to $\mathsf{F}_{\mathsf{BA}}(\kappa)$.

It turns out that the n-universal model is enough to reconstruct $F_{HA}(n)$.

By identifying $F_{HA}(n)$ with $\mathrm{ClopUp}(X_n)$, we can consider the Heyting homomorphism $e\colon F_{HA}(n) \to \mathrm{Up}(U_n)$ that maps $V \in \mathrm{ClopUp}(X_n)$ to $V \cap U_n$.

Theorem

$e\colon F_{HA}(n) \to \mathrm{Up}(U_n)$ *is an embedding and its image is the subalgebra of* $\mathrm{Up}(U_n)$ *generated by the upsets* $\{x : i \in c(x)\}$ *for* $i = 1, \ldots, n$.

Proof (sketch): Since HA is generated by the class of finite Heyting algebras, an equation holds in $F_{HA}(n)$ iff it holds in all the finite n-generated Heyting algebras. It follows that e is an embedding because $\mathrm{Up}(U_n)$ encodes the information about the n-generated finite Heyting algebras.

- $\mathrm{Up}(U_1)$ is isomorphic to $\mathsf{F}_{\mathsf{HA}}(1)$.
- However, when $n > 1$, the algebra $\mathrm{Up}(U_n)$ is uncountable, while $\mathsf{F}_{\mathsf{HA}}(n)$ is not.

So, what's $\mathrm{Up}(U_n)$? It is isomorphic to the profinite completion of $\mathsf{F}_{\mathsf{HA}}(n)$.

The profinite completion of an algebra A is the inverse limit of the system consisting of the finite quotients of A.

Theorem (G. Bezhanishvili-Gehrke-Mines-Morandi 2006)

The profinite completion of a Heyting algebra H dual to an Esakia space X is isomorphic to $\mathrm{Up}(X_{fin})$, where $X_{fin} = \{x \in X : {\uparrow}x \text{ is finite}\}$.

Recall that the points of the n-universal model correspond exactly to the points of X_n whose upset is finite.

There's a different approach to build free Heyting algebras called the step-by-step construction. It was developed for finitely generated free Heyting algebras by Ghilardi (1992) who generalized some results of Urquhart (1973). It has been recently generalized to infinitely many free generators by Almeida (2024).

The idea is to start with the 0-th step $\mathsf{F}_{\mathsf{DL}}(\kappa)$ and think of it as the collection of (equivalence classes of) intuitionistic propositional formulas over κ variables that contain no implications.

Layers of implications are added step by step: at the *n*-th step we obtain the distributive lattice consisting of (the equivalence classes of) all formulas with at most *n* nested implications.

Then $\mathsf{F}_{\mathsf{HA}}(\kappa)$ is obtained as the directed limit of this chain of distributive lattices. When κ is finite all these distributive lattices are finite, but the limit isn't.

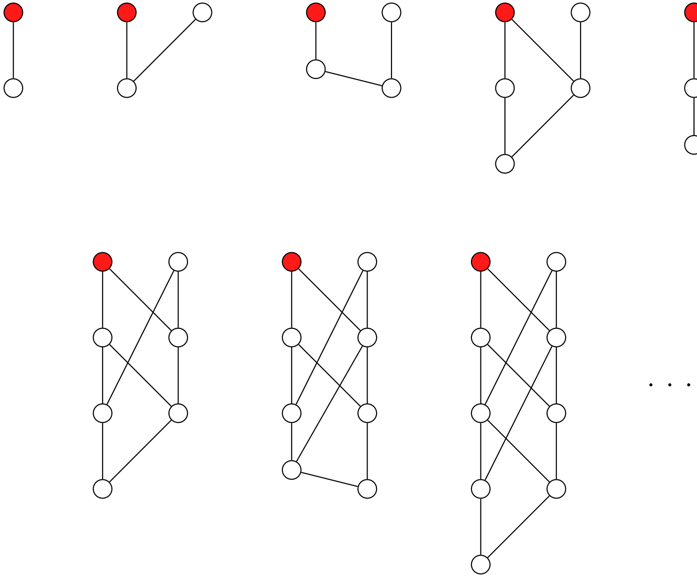
Dually, we start with 2^κ and create a new Priestley space at each step whose points are particular closed subsets of the previous step.

This construction allowed Ghilardi to prove the following surprising fact:

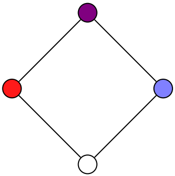
Theorem

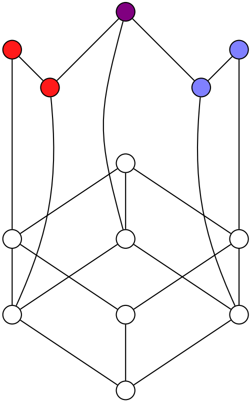
$\mathsf{F}_{\mathsf{HA}}(n)$ *is a bi-Heyting algebra; i.e., its order-dual is also a Heyting algebra.*

Step-by-step for X_1

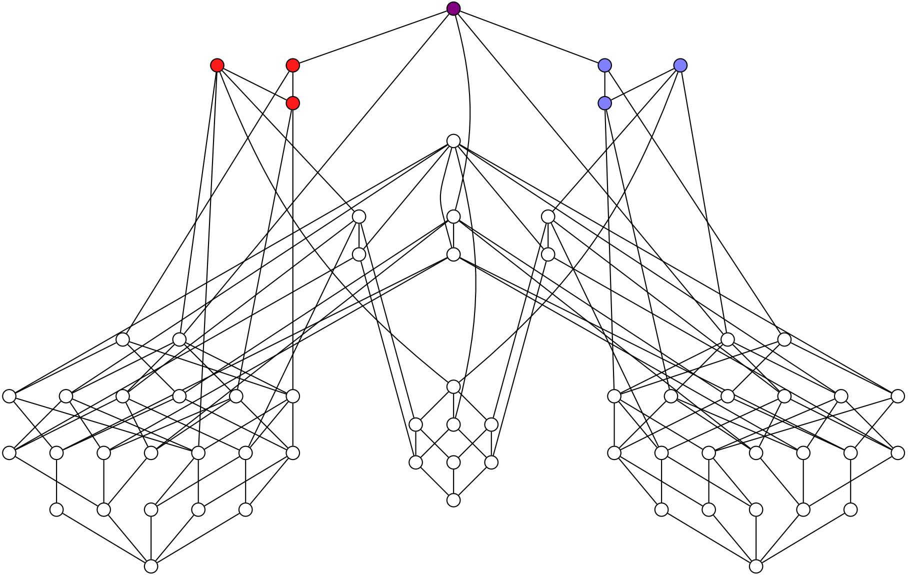


Step-by-step for X_2: zeroth and first steps





Step-by-step for X_2: **part of** the second step



Where to go from here? Some further readings

Free algebras in subvarieties of HA:

- Local finiteness (G. Bezhanishvili-Grigolia 2005, Hyttinen-Martins-Moraschini-Quadrellaro 2025)
- Free Gödel algebras (Grigolia 1987, Aguzzoli-Gerla-Marra 2008, Carai 2024)

Free algebras in varieties of subreducts of Heyting algebras:

- Free pseudocomplemented distributive lattices (Urquhart 1973, Davey-Goldberg 1980)
- Free implicative semilattices (Köhler 1981)

Free algebras in varieties of modal algebras:

- Step-by-step (N. Bezhanishvili-Ghilardi-Jibladze 2014)
- Universal models for Grz, S4, S4.3, GL (Grigolia 1983, 1987)
- Free S4.3-algebras (Esakia-Grigolia 1975)
- Free GL-algebras (van Gool 2014)

Thank you for your attention

References

[AGM08] S. Aguzzoli, B. Gerla, and V. Marra. Gödel algebras free over finite distributive lattices. *Ann. Pure Appl. Logic*, 155(3):183–193, 2008.

[Alm24] R. N. Almeida. Colimits of Heyting algebras through Esakia duality. Available online at `https://arxiv.org/abs/2402.08058v2`, 2024.

[BdJ05] N. Bezhanishvili and D. de Jongh. Intuitionistic logic, 2005. Lecture Notes presented at the ESSLLI, Edinburgh. Available at `https://eprints.illc.uva.nl/id/eprint/200/1/PP-2006-25.text.pdf`.

[Bel86] F. Bellissima. Finitely generated free Heyting algebras. *J. Symb. Logic*, 51(1):152–165, 1986.

[Bez06] N. Bezhanishvili. Lattices of intermediate and cylindric modal logics, 2006. Available at `https://hdl.handle.net/11245/1.262305`.

[BG05] G. Bezhanishvili and R. Grigolia. Locally finite varieties of Heyting algebras. *Algebra Universalis*, 54(4):465–473, 2005.

[BGJ14] N. Bezhanishvili, S. Ghilardi, and M. Jibladze. Free modal algebras revisited: The step-by-step method. In G. Bezhanishvili, editor, *Leo Esakia on Duality in Modal and Intuitionistic Logics*, volume 4 of *Outstanding Contributions to Logic*, pages 43–62. Springer, 2014.

[BGMM06] G. Bezhanishvili, M. Gehrke, R. Mines, and P. J. Morandi. Profinite completions and canonical extensions of Heyting algebras. *Order*, 23(2-3):143–161, 2006.

[Car24] L. Carai. Free algebras and coproducts in varieties of Gödel algebras. Available at arXiv:2406.05480, 2024.

[DG80] B. A. Davey and M. S. Goldberg. The free p-algebra generated by a distributive lattice. *Algebra Universalis*, 11(1):90–100, 1980.

[EG75] L. Esakia and R. Grigolia. Christmas trees. On free cyclic algebras in some varieties of closure algebras. *Polish Acad. Sci. Inst. Philos. Sociol. Bull. Sect. Logic*, 4(3):95–102, 1975.

[EG77] L. Esakia and R. Grigolia. The criterion of Brouwerian and closure algebras to be finitely generated. *Polish Acad. Sci. Inst. Philos. Sociol. Bull. Sect. Logic*, 6(2):46–52, 1977.

[Esa74] L. Esakia. Topological Kripke models. *Dokl. Akad. Nauk SSSR*, 214:298–301, 1974.

[Esa19] L. Esakia. *Heyting algebras. Duality theory*, volume 50 of *Translated from the Russian by A. Evseev. Edited by G. Bezhanishvili and W. Holliday. Trends in Logic*. Springer, 2019.

[Ghi92] S. Ghilardi. Free Heyting algebras as bi-Heyting algebras. *C. R. Math. Rep. Acad. Sci. Canada*, 14(6):240–244, 1992.

[Gri83] R. Grigolia. Finitely generated free s4.3-algebras *(Russian)*. In *Studies on Non-Classical Logics and Formal Systems*, pages 281–286. Nauka, 1983.

[Gri87] R. Grigolia. *Free algebras of nonclassical logics* (Russian). "Metsniereba", Tbilisi, 1987.

[Gri06] R. Grigolia. Co-product of Heyting algebras. *Proc. A. Razmadze Math. Inst.*, 140:83–89, 2006.

[HMMQ24] T. Hyttinen, M. Martins, T. Moraschini, and D. Quadrellaro. Strictly n-finite varieties of heyting algebras. *The Journal of Symbolic Logic*, pages 1–16, 2024.

[Köh81] P. Köhler. Brouwerian semilattices. *Trans. Amer. Math. Soc.*, 268(1), 1981.

[Pri70] H. A. Priestley. Representation of distributive lattices by means of ordered stone spaces. *Bull. London Math. Soc.*, 2:186–190, 1970.

[Pri72] H. A. Priestley. Ordered topological spaces and the representation of distributive lattices. *Proc. London Math. Soc. (3)*, 24:507–530, 1972.

[She78] V. B. Shehtman. Reiger-Nishimura lattices. *Dokl. Akad. Nauk SSSR*, 241(6):1288–1291, 1978.

[Urq73a] A. Urquhart. Free distributive pseudo-complemented lattices. *Algebra Universalis*, 3:13–15, 1973.

[Urq73b] A. Urquhart. Free Heyting algebras. *Algebra Universalis*, 3:94–97, 1973.

[vG14] S. J. van Gool. Free algebras for Gödel-Löb provability logic. In *Advances in modal logic. Vol. 10*, pages 217–233. Coll. Publ., London, 2014.