An introduction to the theory of Borel complexity of classification problems

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I. Some context.

Classification

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Usually, one only cares about these objects up to some notion of isomorphism: for instance, two real vector fields of the same dimension are thought of as being "the same"

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Hence we would like the set of invariants, and the map computing the invariants, to be as concrete (*explicit*) as possible.

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EXPLICIT=BOREL

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For our notion of computability to be useful, our objects need to be encoded so as to form a (standard) Borel space.

Polish, Borel and analytic spaces

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- A subset A of a Polish space X is *analytic* if there exists some continuous map f from a Polish space Y to X such that $A = f(Y)$.

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Definition

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This is due to the fundamental fact that a set is Borel iff it is both analytic and coanalytic.

Polish groups

Many equivalence relations appear as the orbit equivalence relation for some group action $\Gamma \curvearrowright X$:

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Examples

Countable groups; locally compact, metrisable groups; S_∞, the group of all permutations of the integers.

II. Borel classification theory.

Codings

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One may code the same objects in various ways; it is conceivable that the coding can have an influence on the complexity of the classification problem. There seems to be some work to do here!

Another example

Example

We can think of any countable group as having underlying set $\mathbb{N}$; the group is then determined by its multiplication table.
Let us define $GROUP \subset \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ as the set of all α such that

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$GROUP$ is Borel in $\{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ and is thus a standard Borel space.

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If E admits a Borel classification then we say that E is smooth (or concretely classifiable).

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Let E, F be two equivalence relations on standard Borel spaces X, Y. One says that E *Borel reduces* to F ($E \leq_B F$) if there exists a Borel map $\varphi\colon X \to Y$ such that

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If $f \colon X \to Y$ is a Borel reduction of E to F, then from a Borel classification of F one obtains a Borel classification of E. More generally this gives us a precise way to articulate the idea that E is simpler than F.

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This relation is bireducible with the Vitali equivalence relation on $\mathbb{R}$: $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$. The argument used in measure theory classes to produce a non-measurable set from a transversal for this relation proves that E_0 is not smooth.

Classification of countable abelian groups of rank 1

Let G be a countable torsion-free abelian group of rank 1 (i.e. a subgroup of $\mathbb{Q}$). For $a \in G$ and p a prime number one defines the p-type of a $t_p(a) \in \mathbb{N} \cup \{\infty\}$ by

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Baer proved that two torsion-free abelian groups are isomorphic iff they have the same type (which gives a relation bireducible to E_0).

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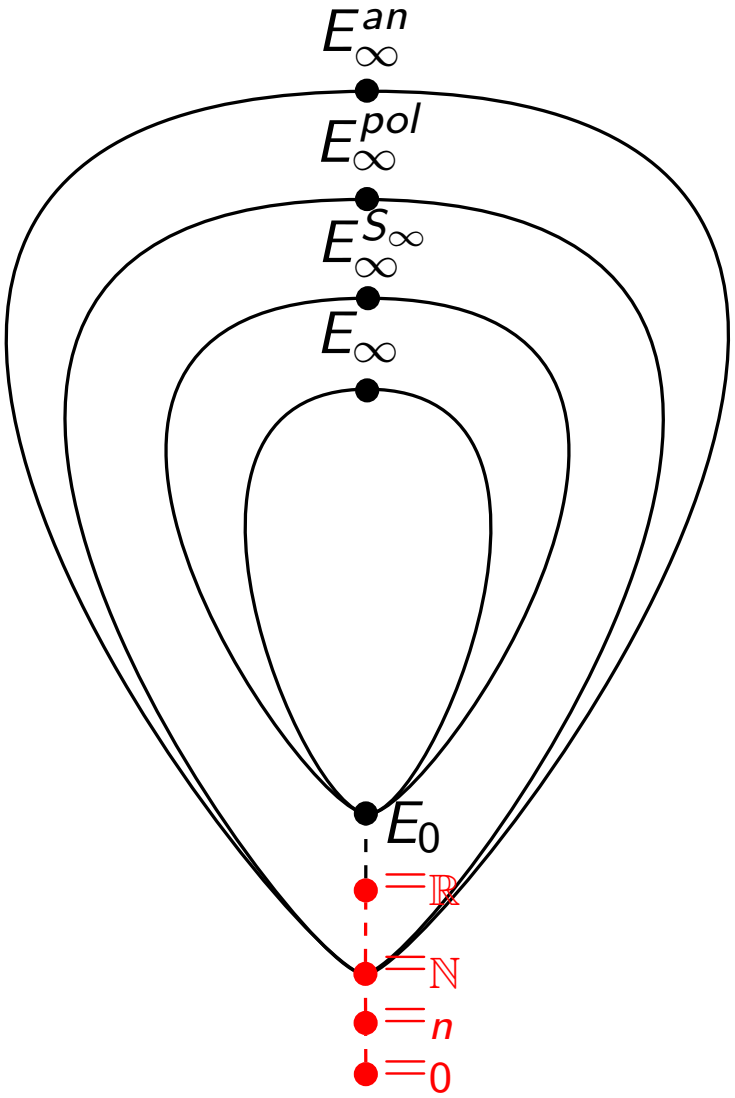
Example

The isomorphism relation beween countable groups, as coded above, is induced by the natural action of S_∞ on the standard Borel space *GROUP*. This relation is analytic non Borel.

A map of the universe.



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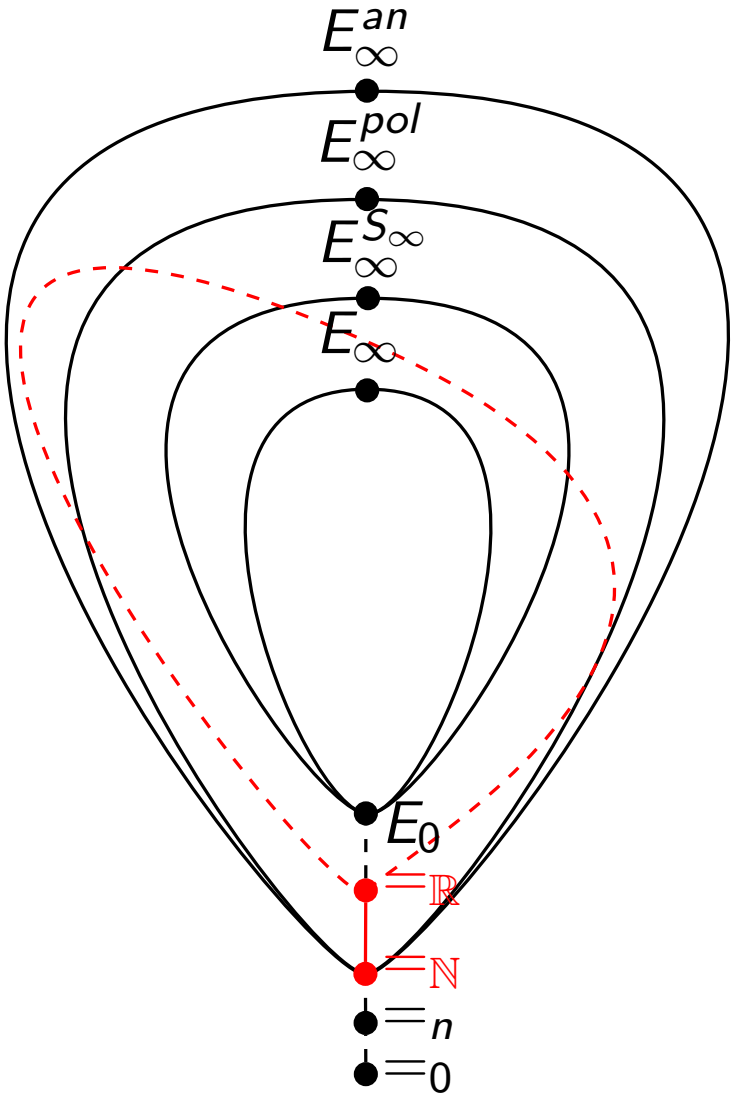

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Theorem (Silver)

Let E be a Borel equivalence relation (even, coanalytic).
Then either $E \leq_B =_{\mathbb{N}}$ or $=_{\mathbb{R}} \leq_B E$.

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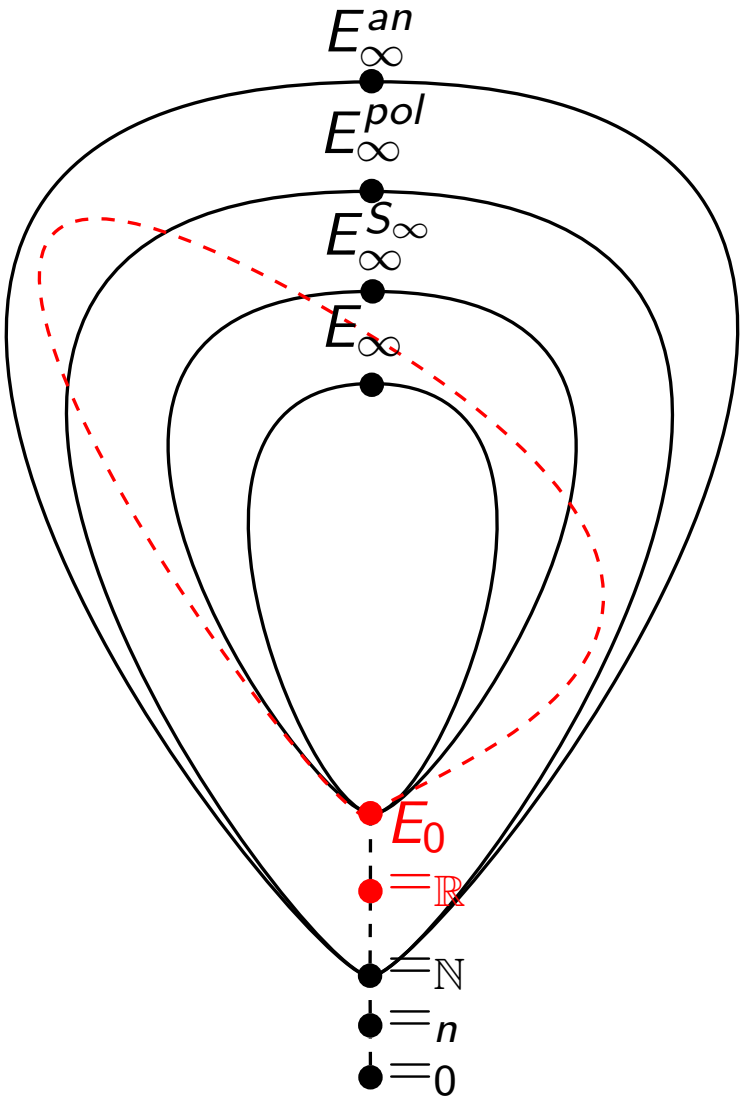

Theorem (Harrington–Kechris–Louveau)

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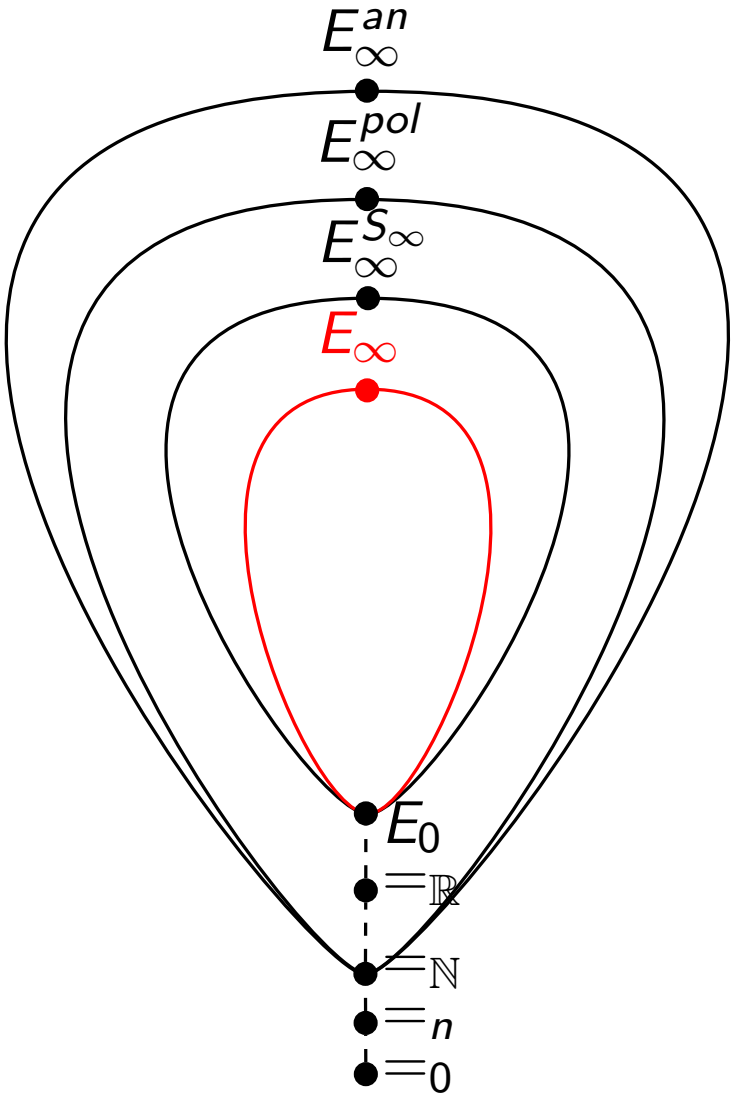

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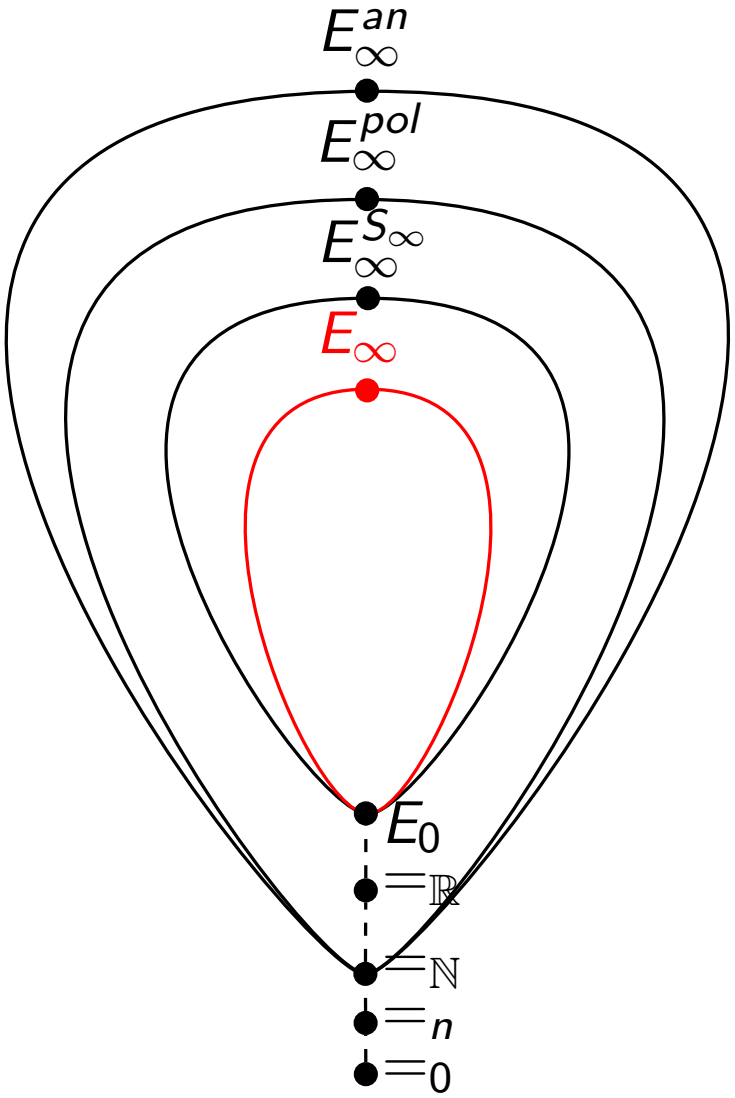

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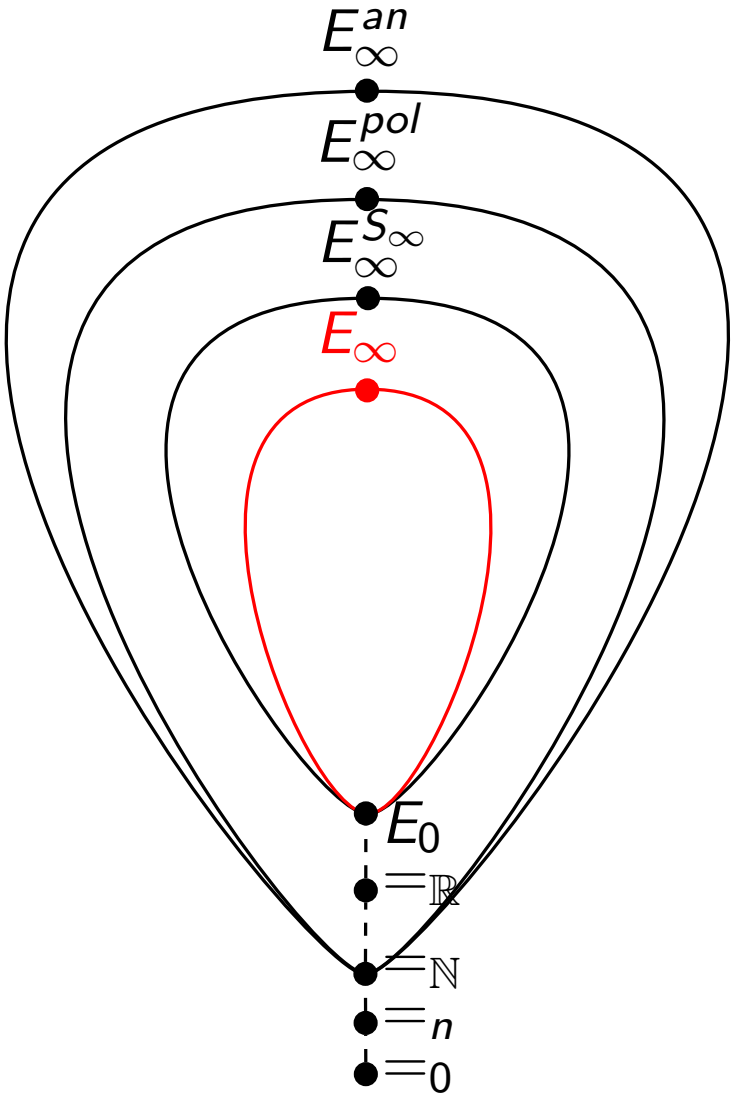

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Example (Dougherty–Jackson–Kechris)

The relation induced by the shift action of F_2 on $\{0,1\}^{F_2}$ is a universal countable Borel equivalence relation.

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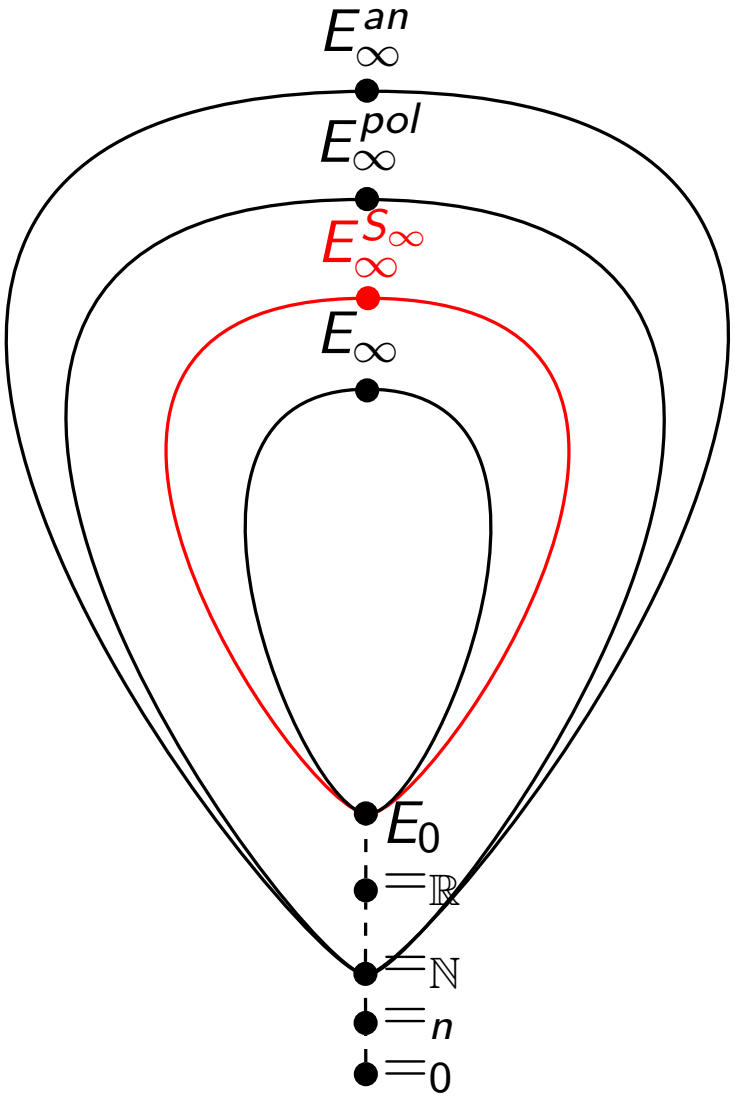

Theorem (Becker–Kechris)

For any Polish group G there exists a universal relation E_∞^G for relations induced by a Borel G-action.

If G is countable, the shift action of G on $(2^{\mathbb{N}})^G$ is $\sim_B E_\infty^G$.

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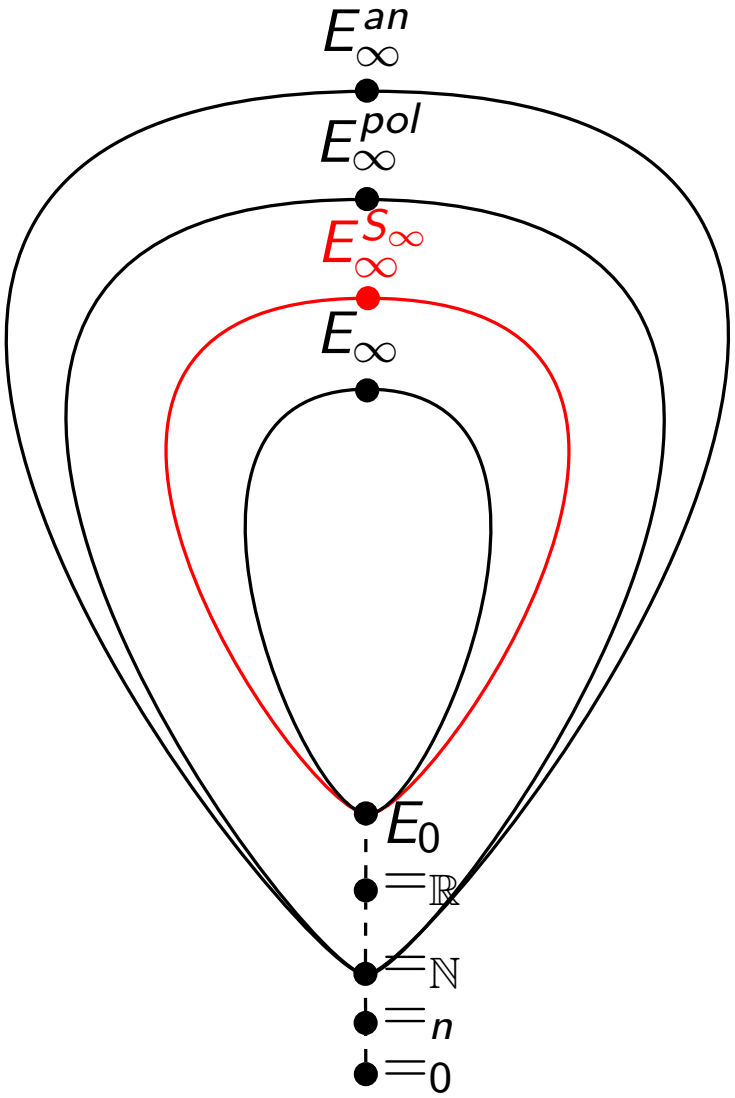

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Example (Friedman–Stanley)

The relation of isomorphism between countable groups (or graphs, or linear orders...) is $\sim_B E_\infty^{S_\infty}$.

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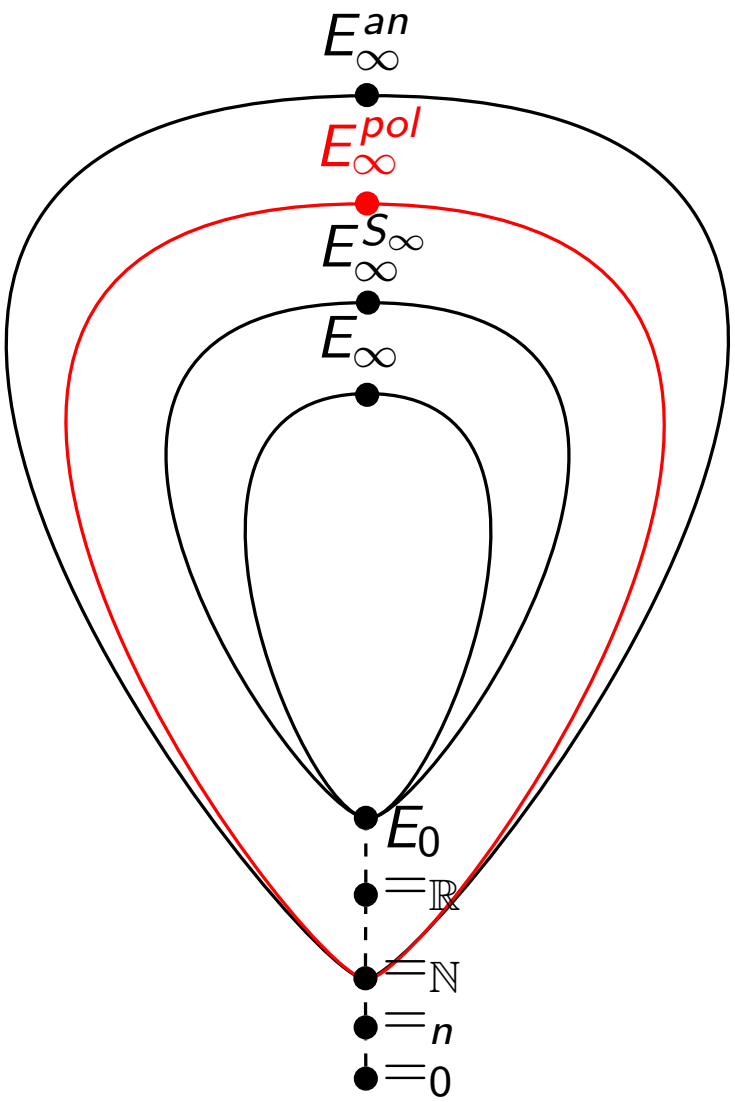

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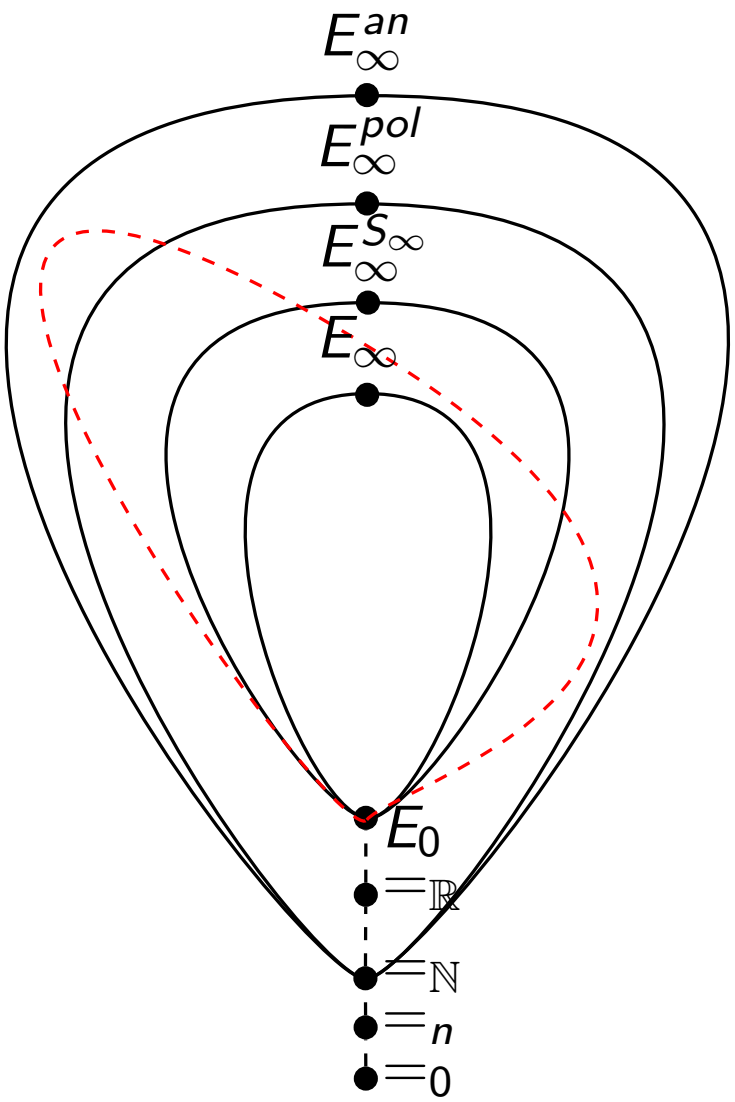

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All Borel equivalence relations do not reduce to such a relation; also, there is no universal Borel equivalence relation.

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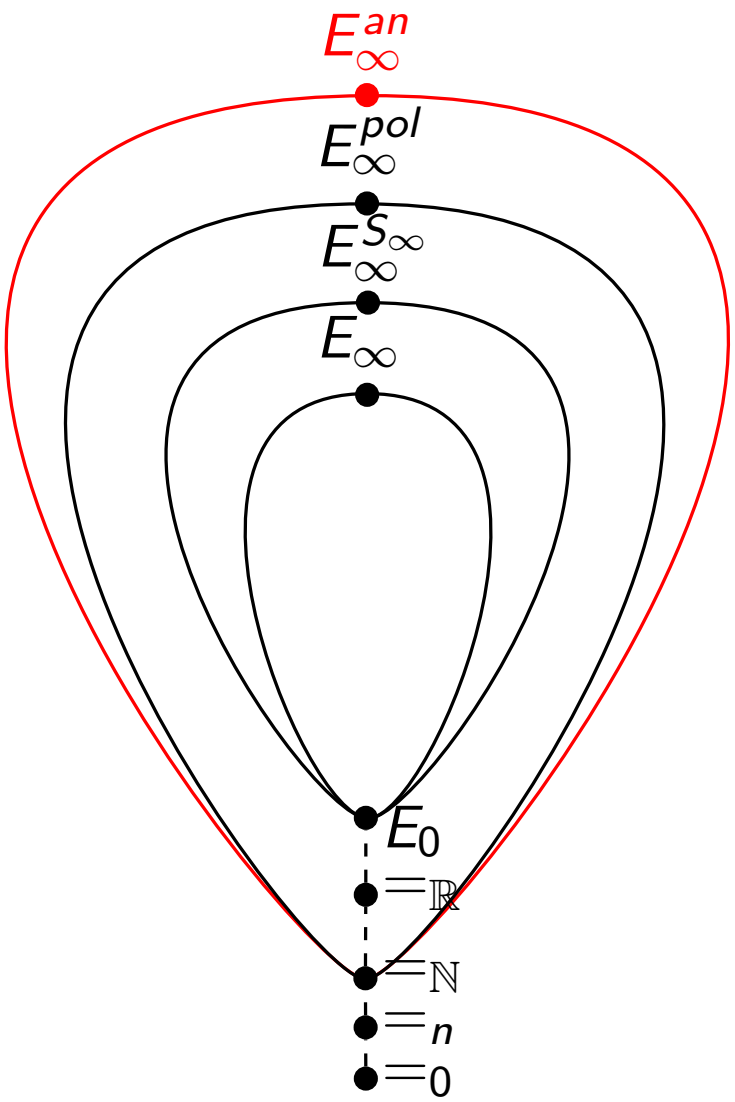

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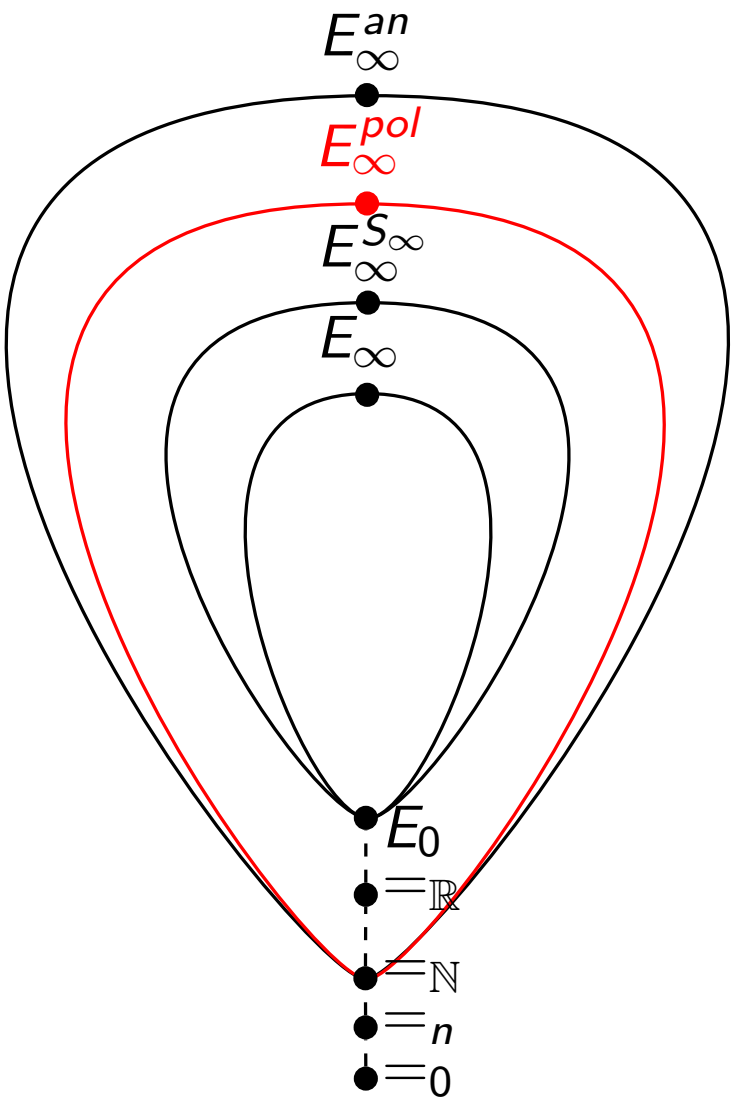

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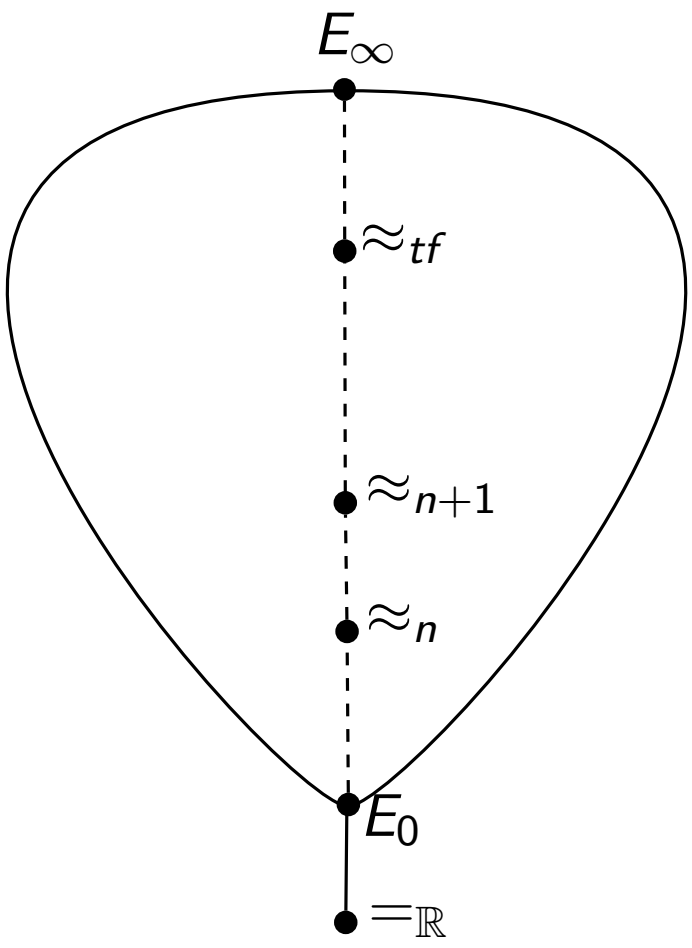
Example

The isometry relation between Polish metric spaces is $\sim_b E_\infty^{pol}$ (Gao-Kechris); same for isometry between separable Banach spaces (M.).

Countable Borel equivalence relations



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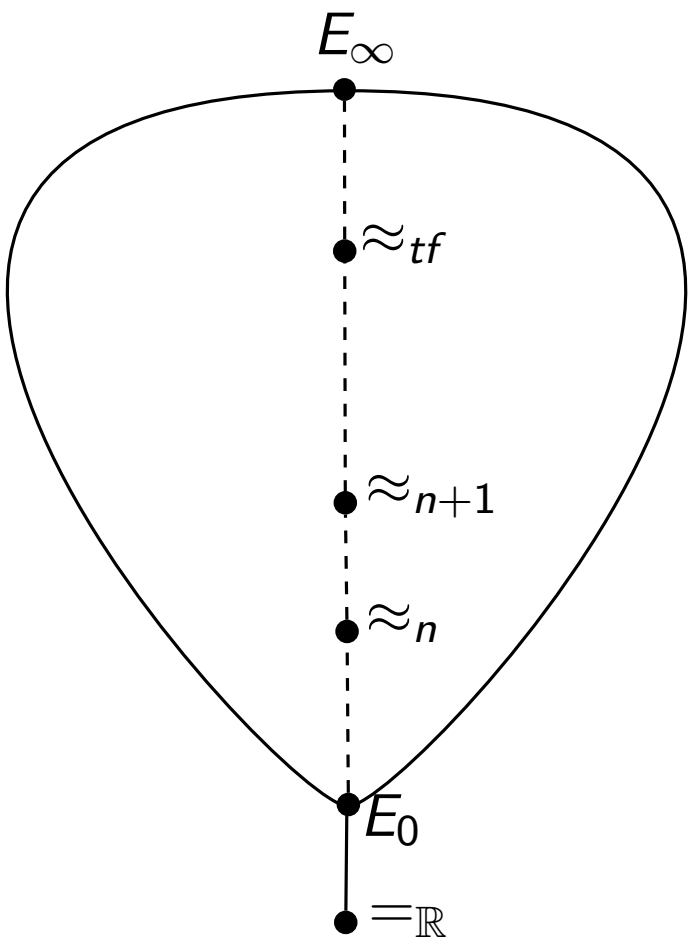

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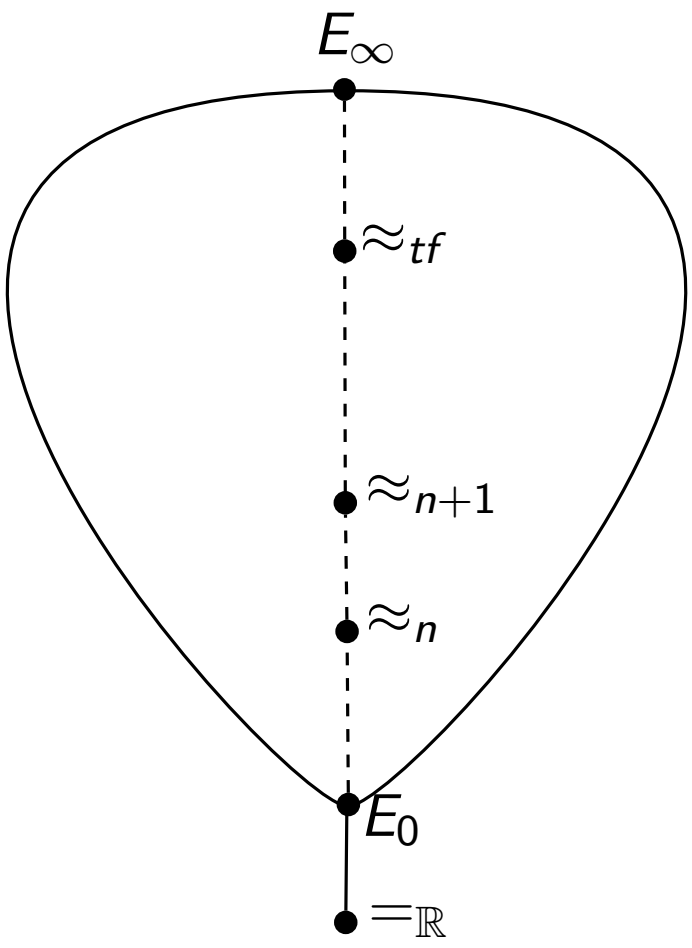
Theorem (Dougherty–Jackson–Kechris)

Let E be a countable Borel equivalence relation. Then $E \leq_B E_0$ iff E is induced by a Borel action of $\mathbb{Z}$.

Countable Borel equivalence relations



Theorem (Feldman–Moore)

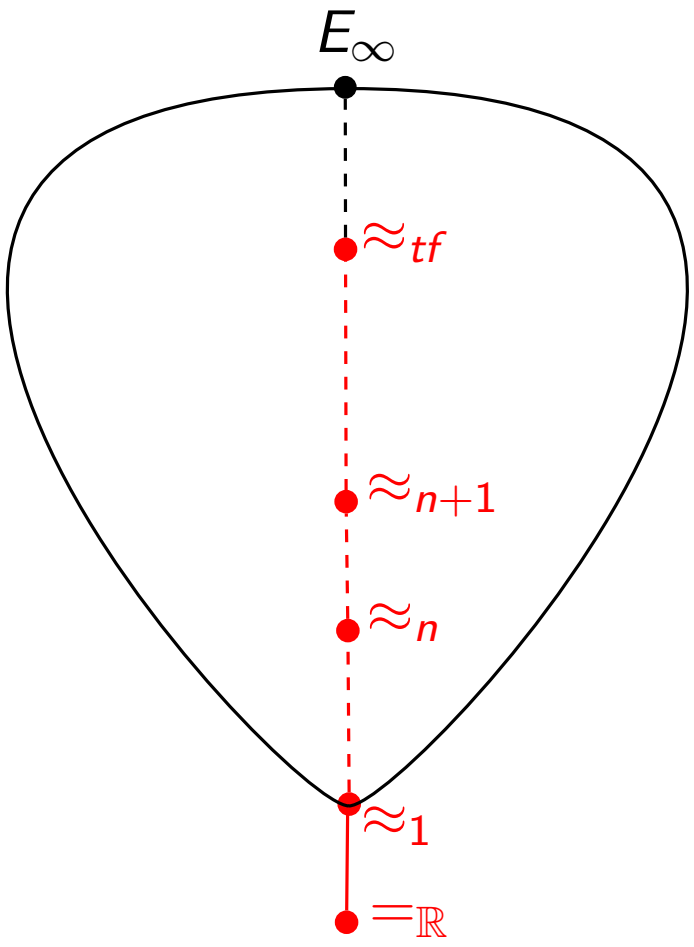
Any countable Borel equivalence relation is induced by a Borel action of a countable (discrete) group G.

Theorem (Dougherty–Jackson–Kechris)

Let E be a countable Borel equivalence relation. Then $E \leq_B E_0$ iff E is induced by a Borel action of $\mathbb{Z}$.

Improved to $\mathbb{Z}^n$ (Weiss) then abelian (Gao–Jackson) then locally nilpotent (Seward–Schneider); open for amenable.

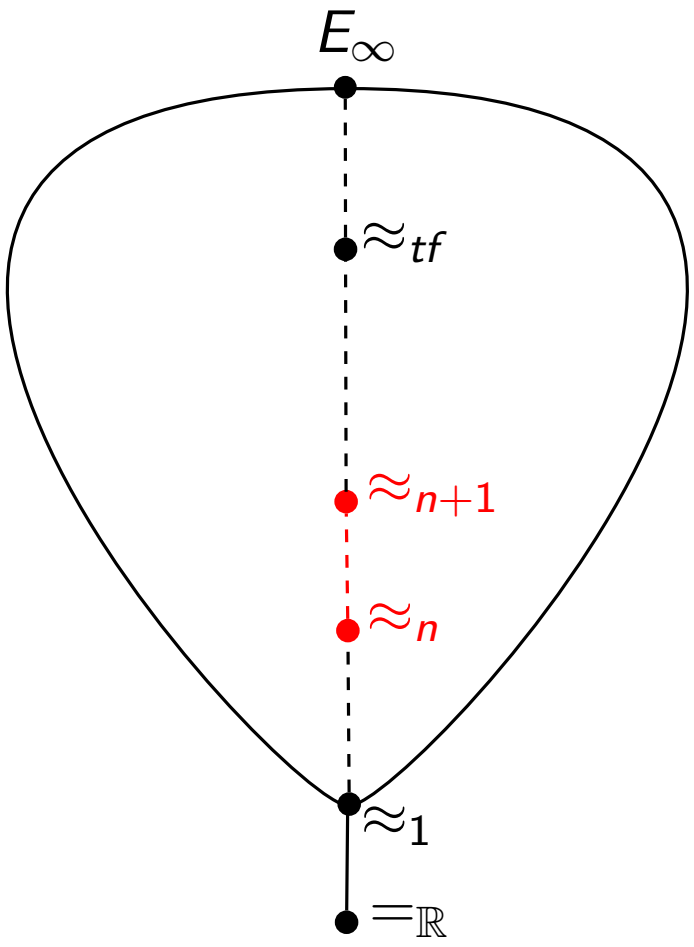
Countable Borel equivalence relations



Example

The relation $\approx_n$ of isomorphism between torsion-free abelian groups of rank $\leq n$ is countable Borel.

Countable Borel equivalence relations



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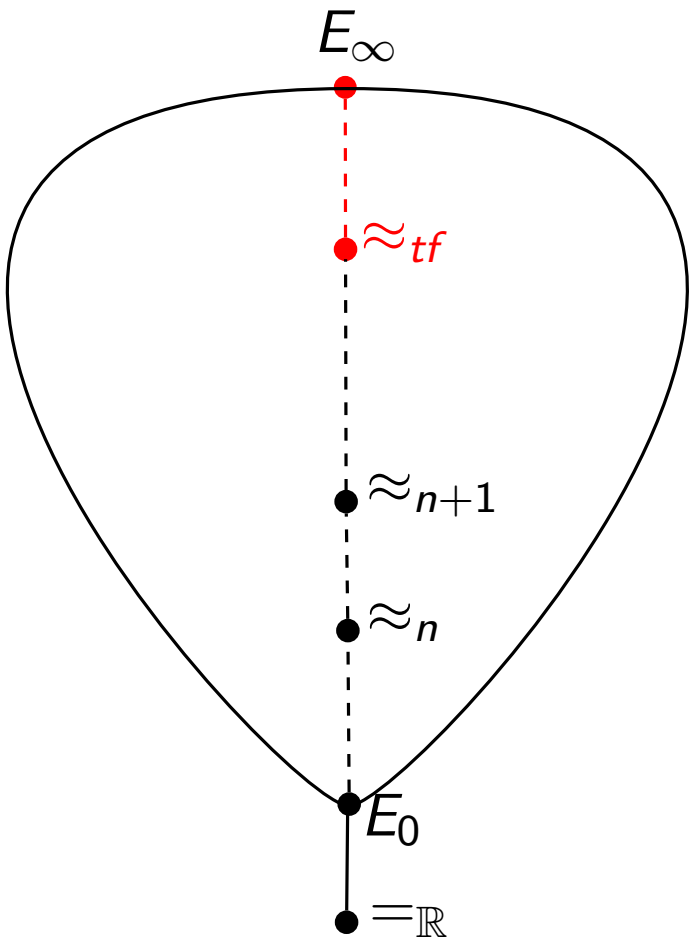
For all n one has $\approx_n <_B \approx_{n+1}$.

Countable Borel equivalence relations



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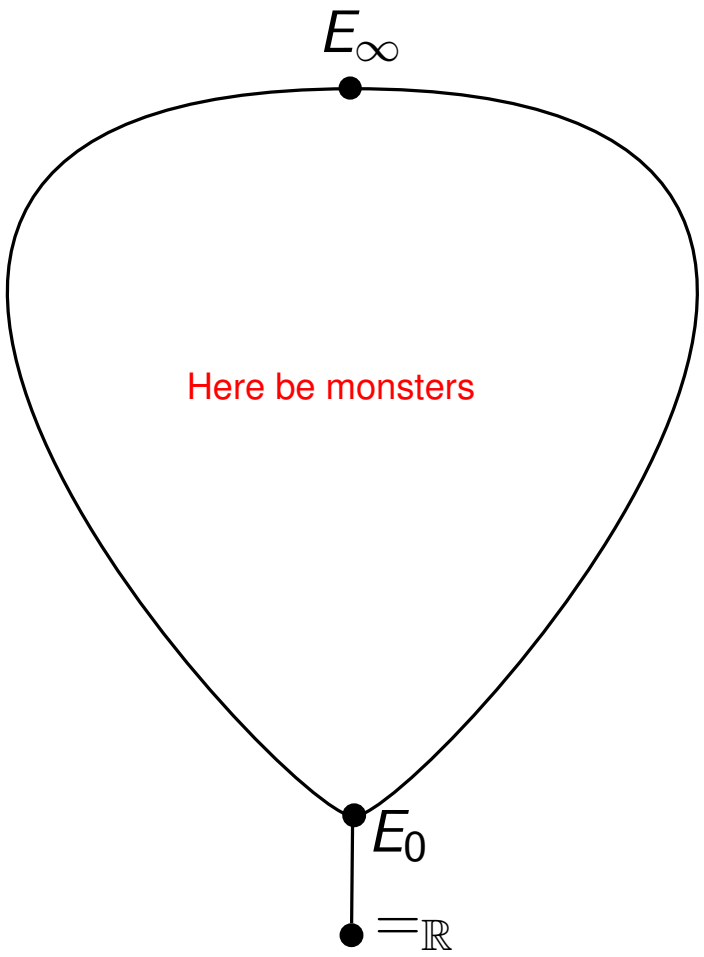
Theorem (Thomas)

The relation $\approx_{tf}$ is not universal for countable Borel equivalence relations.

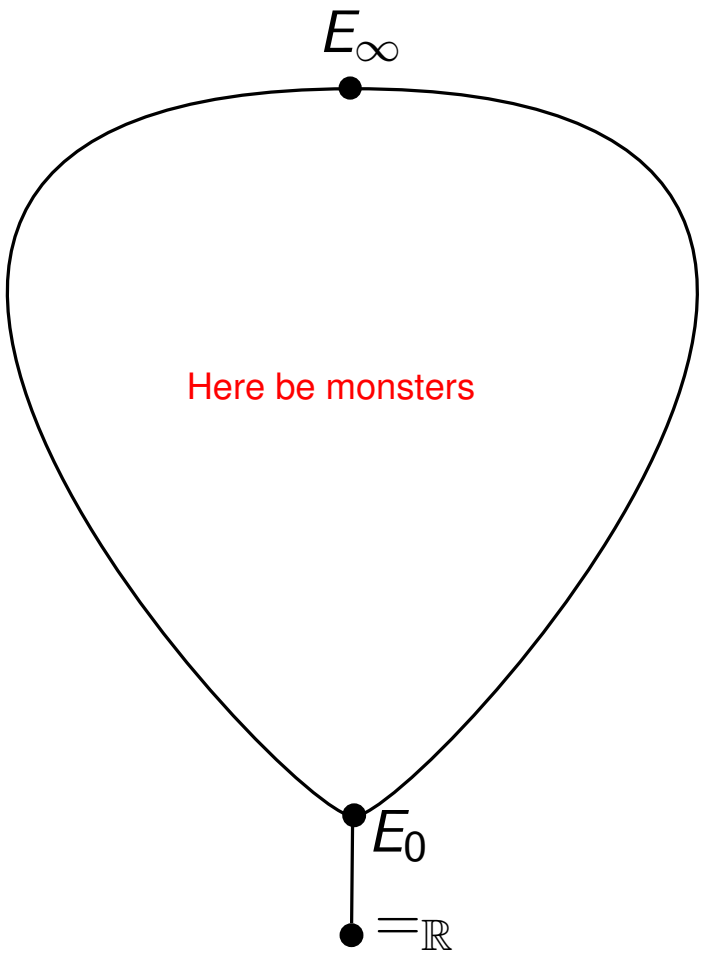
Countable Borel equivalence relations



Theorem (Adams–Kechris)

There exists an order-preserving map from $(\mathcal{P}(N), \subseteq)$ to countable Borel equivalence relations with $\leq_B$.

Countable Borel equivalence relations



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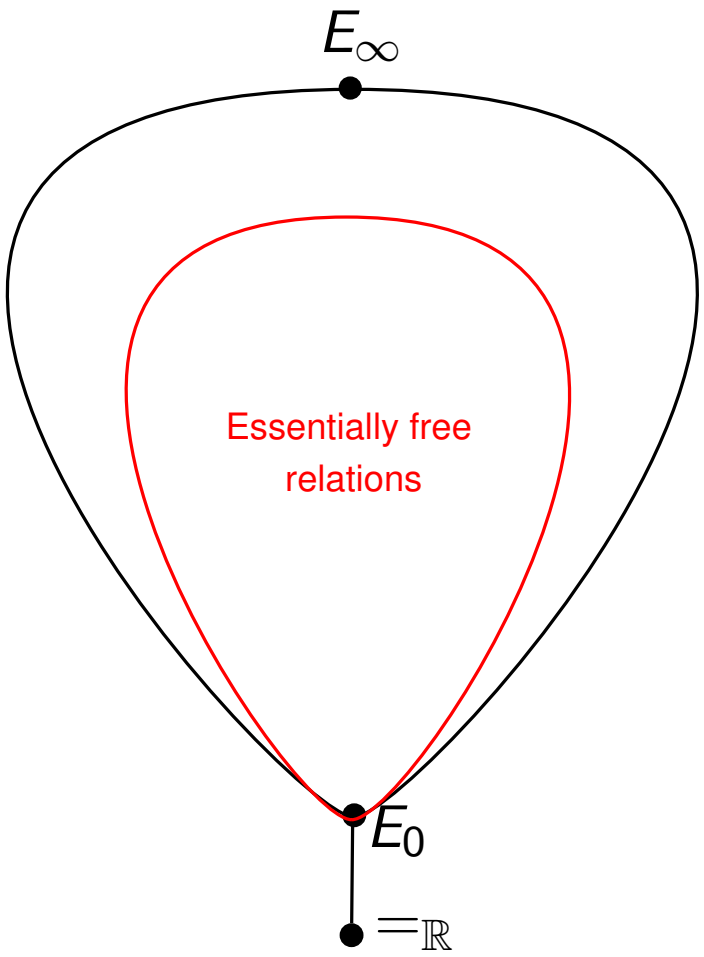
Not much is known about the partial ordering there (for instance, existence of relations with an immediate successor besides $=_{\mathbb{R}}$?).

Countable Borel equivalence relations



Theorem (Thomas)

There exist countable Borel equivalence relations which do not reduce to a relation induced by a *free* action of a countable group.

The Vaught conjecture(s)

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Assume E is induced by a Borel action of S_∞. Is it true that E has either countably many or continuum many classes?

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The Vaught conjecture(s)

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The same question is open in general for Polish groups. Of course it is trivial in a universe where the continuum hypothesis holds, which is not the case of the following variant.

Question

Let E be induced by a Borel action of a Polish group. Is it true that either $E \leq_B =_{\mathbb{N}}$ or $=_{\mathbb{R}} \leq_B E$?

Groups, as men, shall be known by their actions

Question

Assume that G is a Polish group such that the universal equivalence relation induced by a Borel G-action is universal for relations induced by a Polish group action. Must G be a universal Polish group?

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Note: already a very interesting (and probably very difficult) problem for the unitary group of a separable Hilbert space - how to prove that is universal equivalence relation is *not* universal for Polish group actions?

Thank you for your attention!