Topological embeddability between functions.

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Framework

- $f : X \to Y$ means that f is a function, $\mathrm{dom}(f) = X$ and $\mathrm{Im}(f) \subseteq Y$.
- Unless explicitely specified, all spaces are Polish and 0-dimensional.
- Unless explicitely specified, all functions are **Borel**, so preimages of open sets are Borel.
- A function is **Baire class** α if preimages of open sets are $\mathbf{\Sigma}^0_{\alpha+1}$.

The main definition: Solecki's topological embeddability

X, X', Y, Y' topological spaces, $f : X \to Y$ and $g : X' \to Y'$

Definition

A **topological embedding** from f to g is a pair $(\sigma : X \to X', \tau : \mathrm{Im}(f) \to Y')$ of continuous embeddings such that $\tau \circ f = g \circ \sigma$.

Note $f \sqsubseteq g$ when f embeds in g.

$$\begin{tikzcd} {} \arrow[r, "\sigma"] \arrow[d, "f"'] & {} \arrow[d, "g"] \\ {} \arrow[r, "\tau"] & {} \end{tikzcd}$$

Observations

- Topological embedding between functions is a quasi-order, that is, a transitive and reflexive relation.
- If f embeds in g and g is Baire class α, so is f. Take indeed (σ, τ) an embedding and note that $f = \tau^{-1} g \sigma$.
- This is **not** the case if we look at embeddability between graphs of functions, since there are functions of arbitrary Baire class with closed graphs.

Definition

A set A is a **basis** for a class Γ of functions if every function in Γ embeds some element of A.

Some classes of functions admit **finite bases**.

A basis for Borel functions

Note c_X a constant function with domain a space X, Id_X the identity function on X.

Proposition

$\{c_{\mathbb{N}^\mathbb{N}}, \mathrm{Id}_{\mathbb{N}^\mathbb{N}}\}$ *is a basis for all Borel functions on the Baire space.*

Proof. Take $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ Borel. As f is continuous on a dense $\mathbf{\Pi}^0_2$ set, by passing to a subfunction we can suppose that f is continuous.
If f is constant on an open set, then $c_{\mathbb{N}^\mathbb{N}}$ embeds in f.
Otherwise f is injective on a perfect compact subset K of its domain, so $f|_K$ is an embedding. Since K is perfect take an embedding $\sigma : \mathbb{N}^\mathbb{N} \to K$, then $(\sigma, f \circ \sigma)$ is an embedding from $\mathrm{Id}_{\mathbb{N}^\mathbb{N}}$ to f. $\square$

Bases for non-continuous functions

Name $f_0 : \omega + 1 \to 2$ the characteristic function of ω, and $f_1 : \omega + 1 \to \omega$ an injection.

Fact

$\{f_0, f_1\}$ *is a basis for non-continuous functions.*

Proof. Take $x_n \to x$ with $f(x_n) \not\to f(x)$. Wlog $(x_n)_n$ is either constant, then $f_0 \sqsubseteq f$; or it is injective and then $f_1 \sqsubseteq f$. $\square$

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Note P the infinite product $(f_1)^\omega : (\omega + 1)^\omega \to \omega^\omega$. A function is σ-**continuous** is it can be covered by continuous functions with Borel domains.

Theorem (Solecki, Pawlikovski-Sabok)

$\{P\}$ *is a basis for Borel non σ-continuous functions on* $\mathbb{N}^\mathbb{N}$.

This was the motivation for introducing topological embeddability between functions.

Basis for non Baire class 1 functions

Fix $d : \mathbb{Q} \to \mathbb{N}$ any bijection.

Theorem (with Ben Miller)

$\{c_\mathbb{Q}, d, \mathrm{Id}_\mathbb{Q}\}$ *is a basis for* **all** *functions on* $\mathbb{Q}$.

if $f : X \to \mathbb{N}^\mathbb{N}$ and $g : X' \to \mathbb{N}^\mathbb{N}$ have disjoint domains, note

$$f \sqcup g : X \cup X' \to \mathbb{N}^\mathbb{N}$$
$$x \mapsto \begin{cases} 0^\frown f(x) & \text{if } x \in X \\ 1^\frown g(x) \text{otherwise.} \end{cases}$$

There is a 6-element basis for non Baire class 1 functions.

Theorem (with Ben Miller)

$\{\varphi \sqcup \psi \mid \varphi = c_{\mathbb{N}^\mathbb{N}}, \mathrm{Id}_{\mathbb{N}^\mathbb{N}} \wedge \psi = c_\mathbb{Q}, d, \mathrm{Id}_\mathbb{Q}\}$ *is a basis for non Baire class 1 functions on* $\mathbb{N}^\mathbb{N}$.

What about maximal functions?

There is a maximal continuous function!

Let $\pi : \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be the projection on the second coordinate.
When $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is continuous, $\mathsf{Id}_{\mathbb{N}^\mathbb{N}} \times f$ is an embedding.
So $(\mathsf{Id}_{\mathbb{N}^\mathbb{N}} \times f, \mathsf{Id})$ is an embedding from f in π.

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Encouraging, unfortunately...

Theorem (with Yann Pequignot and Zoltan Vidnyanszky)

No Baire class α admits a maximal element, for countable $\alpha \neq 0$.

Idea: Use a generalisation of the Bourgain rank (due to Elekes-Kiss-Vidnyanszky) and prove that embeddability respects this rank.

Is there always a finite basis?

- Getting back to bases results, one can wonder if every upward-closed class of functions admits a finite basis.
- This is equivalent to being a **well-quasi-order**, or **wqo**.
- A quasi-order is a wqo if every subset has minimal elements, and there are no infinite antichains.

Is topological embeddability a wqo on Borel functions?

But once again:

Fact

There is an infinite antichain among continuous functions.

How bad does it fail?

Let's measure the complexity of this quasi-order.
On the space of continuous functions $X \to Y$ we put the **compact-open topology**, generated by

$$S_{X,Y}(K, U) = \{f \in C(X, Y) \mid f(K) \subseteq U\},$$

for $K \subseteq X$ compact and $U \subseteq Y$ open.
If X is compact Polish and Y is Polish, it is a Polish topology.

Theorem (with Yann Pequignot and Zoltan Vidnyanszky)

If X is compact, Polish, 0-dimensional with infinitely many limit points, and if Y is Polish, 0-dimensional and not discrete then $(C(X, Y), \sqsubseteq)$ is a $\mathbf{\Sigma}^1_1$-complete quasi-order.

A dichotomy

Theorem (with Yann Pequignot and Zoltan Vidnyanszky)

If X has infinitely many limit points, and if Y is not discrete then $(C(X,Y), \sqsubseteq)$ is a $\mathbf{\Sigma}^1_1$-complete quasi-order.

So, in these cases, topological embeddability reduces every Borel quasi-order, so it is as far from being a wqo as possible..
What about the other cases?
It turns out to be wqo!

Theorem (with Yann Pequignot and Zoltan Vidnyanszky)

If X and Y are Polish 0-dimensional and X is compact then

- *either $(C(X,Y), \sqsubseteq)$ is a $\mathbf{\Sigma}^1_1$-complete quasi-order,*
- *or it is wqo.*

An infinite antichain

Given $n \geq 2$ define a function f_n:

$$
\begin{aligned}
f_n : n \times (\omega + 1) &\longrightarrow (n \times \omega) + 1 := (n \times \omega) \cup \{\infty\} \\
(i, \omega) &\longmapsto \infty \\
(i, k) &\longmapsto \begin{cases} (i, l) & \text{if } k = 2l \\ (i+1), l) & \text{if } k = 2l + 1 \end{cases}
\end{aligned}
$$

where $i + 1$ is intended modulo n. Take now $m < n$.

- $n \times (\omega + 1)$ does not embed in $m \times (\omega + 1)$, so $f_n \not\sqsubseteq f_m$
- the m-cycle does not embed injectively in the n-cycle, so $f_m \not\sqsubseteq f_n$.

A reduction from graph-embeddability: sketch idea

Following this line of idea, we call C the set of countable graphs on ω with no isolated points, and $\prec$ the quasi-order of injective homomorphism between them.

Proposition (with Yann Pequignot and Zoltan Vidnyanszky)

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Now if Y is not discrete there is an embedding $\iota_Y : \omega + 1 \to Y$. And if X has infinitely many limit points one can build a specific continuous surjection $\rho_X : X \to \omega^2$ such that

Proposition (with Yann Pequignot and Zoltan Vidnyanszky)

$G \mapsto \iota_Y \circ \phi(G) \circ \rho_X$ *is a continuous reduction from* $(\mathrm{C}, \prec)$ *to* $(C(X, Y), \sqsubseteq)$.

We finally use $\mathbf{\Sigma}^1_1$-completeness of $(\mathrm{C}, \prec)$, proven by Louveau and Rosendal.

Some questions

First, two obvious ones

- Can we have a similar dichotomy outside 0-dimensional spaces?
- Which are the classes of functions admitting finite bases?

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Observe then that if X, X' have infinitely many limit points, and if Y, Y' are not discrete then our dichotomy yields Borel reductions between $C(X, Y)$ and $C(X', Y')$ for free, but..

- When is there a **continuous** reduction between $C(X, Y)$ and $C(X', Y')$?
- If there is a continuous reduction, when is there a topological embedding?

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Thank you!