When Borel reducibility is not enough...

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Borel Reducibility of Equivalence Relations
Lausanne, 29.05.2017

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For example, when X carries a nice Borel structure (e.g. a Borel structure induced by a Polish topology), then a quite concrete solution would be a pair (I, φ) where $I = \mathbb{R}$ (equivalently, I is any Polish space) and φ is a Borel function.

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n-square complex matrices are concretely classifiable up to similarity: a solution for this classification problem is the map assigning to each such matrix its canonical Jordan form.

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Now invariants are equivalence classes with respect to some equivalence relation: what does it mean that the assignment map φ is "concrete/simple" in this broader context?

Borel reducibility

Definition

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$E \sim_B F$ ("E and F are **Borel bi-reducible**") iff $E \leq_B F \leq_B E$.

Equivalently, $E \sim_B F$ if and only if there are injections from each quotient space to the other one, both admitting Borel liftings.

Remark

Each quotient space X/E may be naturally equipped with a **quotient Borel structure** by stipulating that $A \subseteq X/E$ is Borel if and only if $\bigcup A = \{x \in X \mid [x]_E \in A\}$ is a Borel subset of X.

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Similar considerations hold for Borel bi-reducibility (which implies, in particular, that the quotient spaces are Borel bi-embeddable).

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- $E \leq_B F$ with F fairly simple $\rightsquigarrow$ **classification results**
- $F \leq_B E$ with F very complicate $\rightsquigarrow$ **anti-classification results**

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- often the space of objects X carries a natural standard Borel structure, but no preferred Polish topology (e.g. the space $X = F(Z)$ of closed subsets of a Polish space Z);
- there are solutions to classification problems commonly accepted in mathematics which are not given by continuous functions: this would lead to the problem of establishing a generally accepted threshold for the notion of "simplicity" (*sorites paradox*).

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In contrast, when we merely know that f witnesses $E \leq_B F$, we just get that the F*-saturation* of the range of f, i.e. the set

$$\{y \in Y \mid y \; F \; f(x) \text{ for some } x \in X\},$$

is analytic but not necessarily Borel.

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Furthermore, since invariants should represent, in a sense, the E-equivalence classes, it would be desiderable to be able to reconstruct (in a simple way) from such invariants the objects to which they are assigned (up to E-equivalence).

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This requirement is so natural that it was already considered by H. Friedman and Stanley in the first paper on Borel reducibility from 1989, where it is called **Borel recovery property**.

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This is apparently finer than Borel bi-reducibility: indeed, $\approx_G \sim_B \approx_H$ yields only two injections $\varphi\colon \widehat{G} \to \widehat{H}$ and $\psi\colon \widehat{H} \to \widehat{G}$ admitting Borel liftings, but it does not imply in general that $\psi = \varphi^{-1}$.

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E is **faithfully Borel reducible** (or **FS-reducible**) to F ($E \leq_{fB} F$) if there is a witness f of $E \leq_B F$ such that for every Borel E-saturated $A \subseteq X$, the F-saturation of $f(A)$ is Borel.

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From $\leq_{fB}$, we can naturally define the induced equivalence relation $\sim_{fB}$ of **faithful Borel bi-reducibility** by setting

$$E \sim_{fB} F \iff E \leq_{fB} F \leq_{fB} E.$$

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Faithful Borel reducibility is strictly finer than $\leq_B$, and in fact it yields a better analysis of the complexity of certain classification problems.

Theorem (H. Friedman-Stanley, 1989; Gao, 1998)

Let $\cong_{\mathsf{GRAPH}}$, $\cong_{\mathsf{LO}}$, and $\cong_{\mathsf{TREE}}$ denote the isomorphism relations on, respectively, countable graphs, countable linear orders, and countable trees. Then

$$\cong_{\mathsf{GRAPH}} \sim_B \cong_{\mathsf{LO}} \sim_B \cong_{\mathsf{TREE}}$$

but

$$\cong_{\mathsf{GRAPH}} \not\leq_{fB} \cong_{\mathsf{LO}} \qquad \text{and} \qquad \cong_{\mathsf{GRAPH}} \not\leq_{fB} \cong_{\mathsf{TREE}}.$$

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Under this interpretation, Gao's result may be seen as a proof of the fact that the theory of (countable) graphs cannot be interpreted in the theory of (countable) linear orders or in the theory of (countable) trees.

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Definition

E is **faithfully Borel reducible** (or **FS-reducible**) to F ($E \leq_{fB} F$) if there is a witness f of $E \leq_B F$ such that for every Borel E-saturated $A \subseteq X$, the F-saturation of $f(A)$ is Borel.

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However, even when $E \leq_{fB} F$ it may be impossible to recover an object from the invariant, i.e. the reducibility may fail to have the *Borel recovery property*. This happens because the requirement in the (equivalent reformulation of the) definition is asymmetric: we demand that f has a Borel lifting, but we don't ask the same for f^{-1}.

Classwise Borel isomorphism and classwise Borel embedding

Recall that $\sim_B$ corresponds, roughly speaking, to Borel bi-embeddability between the quotient spaces.

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Schröder-Bernstein theorem for reducibilities (M.)

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Some features of $\sqsubseteq_{cB}$ and $\simeq_{cB}$

- $\sqsubseteq_{cB}$ fully overcomes both Objection 2 and Objection 3 when used as a classification tool: we can recognize in a Borel way the invariants used in the classification, and it has the Borel recovery property.

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- $\sqsubseteq_{cB}$ is the only reducibility admitting a "Schröder-Bernstein theorem".
- If $E \sqsubseteq_{cB} F$, then $E \leq_{fB} F$. Thus also $\sqsubseteq_{cB}$ is strictly finer than $\leq_B$: indeed $\cong_{\mathsf{GRAPH}} \sim_B \cong_{\mathsf{LO}}$ but $\cong_{\mathsf{GRAPH}} \not\sqsubseteq_{cB} \cong_{\mathsf{LO}}$ (and the same for $\cong_{\mathsf{TREE}}$).

A comparison in the Borel realm

Theorem (Friedman-M.)

There is $E \sim_B \mathrm{id}(\mathbb{R})$ such that $E \not\sqsubseteq_{cB} \mathrm{id}(\mathbb{R})$ (hence also $E \not\sim_{cB} \mathrm{id}(\mathbb{R})$).

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Recall that: $E \sqsubseteq_{cB} F$ implies $E \leq_{fB} F$, which in turn implies $E \leq_B F$.

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Let E be an orbit equivalence relation and F be a Borel equivalence relation. If $E \leq_B F$, then $E \leq_{fB} F$, and in fact also $E \sqsubseteq_{cB} F$.

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Let E, F be Borel orbit equivalence relations. Then
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Remark: By the previous counterexample, this cannot be extended to arbitrary Borel equivalence relations (this answers a question of Gao from 2001).

An application

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Let's go back to the unitary equivalence relation $\approx_G$ on the space of irreducible representations of a countable group G, and to the associated unitary dual $\widehat{G}$. By definition, $\widehat{G}$ and $\widehat{H}$ are *Borel isomorphic* iff $\approx_G \simeq_{cB} \approx_H$. Therefore

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Thus, in the end, a strengthening of Borel reducibility allowed us to reconcile $\sim_B$ with the notion of Borel isomorphism between unitary duals (Objection 4).

Remark: Recently, this simple observation allowed Simon Thomas to use Borel reducibility to obtain beautiful results pushing further the analysis of unitary duals of non-Abelian-by-finite countable groups.

Structural consequences

Kechris and Macdonald (implicitly) observed that the use of $\sqsubseteq_{cB}$ yields to interesting structural properties for equivalence relations under $\leq_B$.

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Moreover, the same is true if $\mathcal{E} =$ all Borel orbit equivalence relation, or $\mathcal{E} =$ all treeable countable Borel equivalence relations, and so on.

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Remark: This use of cardinality algebras mathematically justifies a further interpretation of Borel reducibility, namely

$E \leq_B F \rightsquigarrow$ the Borel cardinality of X/E is less than or equal to the Borel cardinality of Y/F.

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If F is essentially in $\mathcal{E}$, then F is generically in $\mathcal{E}$.

Other (trivial?) observations

Proposition

Let E be an essentially orbit equivalence relation on a Polish space X. If all countable unions of E-equivalence classes are *not comeager*, then $\mathrm{id}(\mathbb{R}) \leq_B E$.

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Theorem (Mycielski)

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Theorem (Harrington-Kechris-Louveau)

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Beyond Borel orbit eq. rel. ($\mathrm{id}(\mathbb{R})$ and E_0)

Theorem (M.)

Let E be either $\mathrm{id}(\mathbb{R})$ or E_0, and F be an arbitrary orbit equivalence relation. Then $E \leq_B F \iff E \sqsubseteq_{cB} F$.

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This follows from the Crucial lemma and the fact that $E \leq_B E \restriction C$ for any comeager E-saturated C (use the HKL theorem in the case of E_0).

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In parts 3 and 4 we may equivalently replace $\sim_B$ with $\simeq_{cB}$.

A comparison between the various reducibilities

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Proof.

Let $C_i \subseteq \mathbb{R}^2$ (for $i = 1, 2$) be Borel sets with no Borel uniformization and such that $\mathrm{proj}(C_1)$ is Borel, while $\mathrm{proj}(C_2)$ is a proper analytic set.

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Let $C_i \subseteq \mathbb{R}^2$ (for $i = 1, 2$) be Borel sets with no Borel uniformization and such that $\mathrm{proj}(C_1)$ is Borel, while $\mathrm{proj}(C_2)$ is a proper analytic set. Set $E_i = E \oplus F_{C_i}$. $\square$

A comparison between the various reducibilities

Theorem (M.)

For every orbit equivalence relation E satisfying $\mathrm{id}(\mathbb{R}) \leq_B E$ there are E_1, E_2 such that both $E_1 \sim_B E$ and $E_2 \sim_B E$, and moreover:

- $E_1 \leq_{fB} E$ while $E_2 \not\leq_{fB} E$;
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The reducibilities $\leq_B$, $\leq_{fB}$, and $\sqsubseteq_{cB}$ are all distinct.

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Question

Are there $\sim_B$-classes containing large $\sqsubseteq_{cB}$-antichains, or long (ascending or descending) $\sqsubseteq_{cB}$-chains?

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Beyond Borel orbit equivalence relations

Definition (Kechris)

A Borel equivalence relation E is **idealistic** if there is a map assigning to each $C \in X/E$ a nontrivial σ-ideal I_C on C such that $C \mapsto I_C$ is Borel in the following sense: For each Borel $A \subseteq X^2$, the set $A_I \subseteq X$ is Borel, where $x \in A_I \iff \{y \in [x]_E \mid (x,y) \in A\} \in I_{[x]_E}$.

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Let E be idealistic and F be Borel. If $E \leq_B F$, then $E \sqsubseteq_{cB} F$.

Corollary

If E, F are Borel idealistic equivalence relations, then

$$E \leq_B F \iff E \sqsubseteq_{cB} F \qquad \text{and} \qquad E \sim_B F \iff E \simeq_{cB} F.$$

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Remark 2: It is plausible that in the last result one can replace $\leq_{fB}$ with $\sqsubseteq_{cB}$.

Conclusions

Speaking about variants of Borel reducibility, H. Friedman and Stanley wrote in their paper from 1989:

> *We shall rarely work with these notions in this paper, as we conjecture that the resulting notions [of reducibility] are extremely sparse and that there are few positive results to be had.*

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A systematic study comparing all these various reducibilities could turn out to be useful to shed light on some phenomena in the theory of Borel reducibility which remain invisible to a more classical approach.

Thank you for your attention!