Borel completeness of a complete knot invariant

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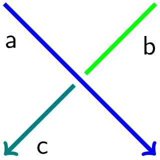
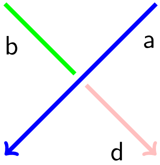
An algebraic structure associated with knots

Take an oriented knot diagram. We define an algebraic structure with two binary relations $*$ and $*'$, a generator for each arc of the diagram, and a relation for each crossing, as follows:

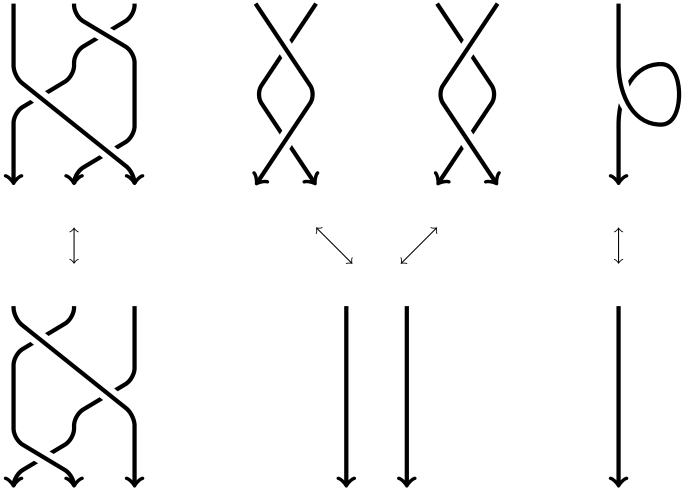


$a*b=c$

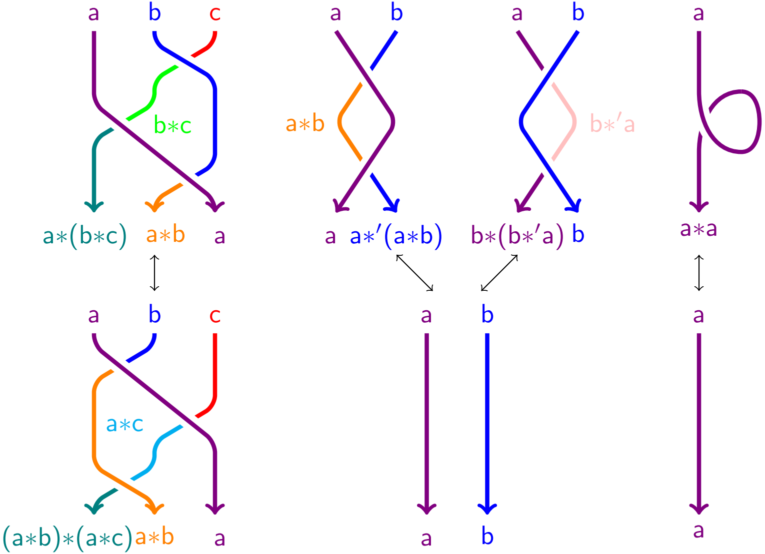


$a*'b=d$

The Reidemeister moves



Respecting the Reidemeister moves



Definitions

A *quandle* is a set with a binary operation $*$ such that

1. $\forall a \forall b \forall c [a * (b * c) = (a * b) * (a * c)]$
2. for all a, the map $b \mapsto a * b$ is a bijection
3. $\forall a [a * a = a]$.

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Example

Any group with the operation of conjugation ($a * b = aba^{-1}$) is a quandle.

Quandles as knot invariants

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Gist of Theorem (A.B.-T., S. Miller)

Isomorphism of general countable quandles is as complex as possible.

Classes of structures

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isomorphism relations on first order classes of countable structures, such as of graphs, groups, or indeed quandles.

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Let L be a countable set of relation symbols. We consider the set $\mathbf{Mod}(L)$ of L-structures with underlying set $\mathbb{N}$. We can view such a structure as being encoded by an element of

$$\prod_{R \in L} 2^{\mathbb{N}^{a(R)}}$$

where $a(R)$ is the arity of R.

Example:

A directed graph G on vertex set $\mathbb{N}$ is determined by a function from $\mathbb{N}^2$ to 2, taking (m, n) to 1 if there is an edge from m to n in G, and to 0 if not.

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Giving $\mathbf{Mod}(L)$ the corresponding topology, we have that an open subbase set is given by a single bit of information. For example, if L_{Gr} is the vocabulary for directed graphs (with a single binary relation),

$$\{G \in \mathbf{Mod}(L_{Gr}) \mid G \text{ doesn't have an edge from 3 to 177}\}$$

is a subbase set.

Functions

If L contains (k-ary) function symbols, they can be represented as ($k+1$-ary) relations.

For a $k+1$-ary relation F to represent a function, it must satisfy

$$\forall m_1 \cdots \forall m_k \forall n_1 \forall n_2 \neq n_1 \neg\big(F(m_1, \ldots, m_k, n_1) \wedge F(m_1, \ldots, m_k, n_2)\big)$$

and

$$\forall m_1 \cdots \forall m_k \exists n(F(m_1, \ldots, m_k, n)).$$

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Axioms

Similarly, any class of structures for the vocabulary L given by countably many first order axioms Th forms a Borel subspace $\mathbf{Mod}(Th)$ of $\mathbf{Mod}(L)$.

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Viewed the other way:

For any bijection g from $\mathbb{N}$ to $\mathbb{N}$ and any $\mathcal{M}$ in $\mathbf{Mod}(L)$, there is an $\mathcal{N}$ in $\mathbf{Mod}(L)$ such that g is an isomorphism from $\mathcal{M}$ to $\mathcal{L}$: define

$$\mathcal{N} \models R(n_1, \ldots, n_k) \quad \longleftrightarrow \quad \mathcal{M} \models R(g^{-1}(n_1), \ldots, g^{-1}(n_k))$$

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Defining $g \cdot \mathcal{M}$ to be this $\mathcal{N}$, we have a group action of the group S_∞ (of permutations of $\mathbb{N}$) on **Mod**(Th) — the *logic action*.

Borel completeness

Definition

We say a first order class $\mathcal{C} = \mathbf{Mod}(Th_{\mathcal{C}})$ of countable L-structures for some L is *Borel complete* if the isomorphism relation of every other such class Borel reduces to its isomorphism relation: for every other first order class of countable structures $\mathcal{D} = \mathbf{Mod}(Th_{\mathcal{D}})$,

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Examples

- Graphs
- Trees
- Linear Orders
- Groups

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We construct a mapping Q taking (directed, irreflexive) graphs to quandles such that

$$\Gamma \cong_{Graphs} \Gamma' \qquad \text{iff} \qquad Q(\Gamma) \cong_{Quandles} Q(\Gamma').$$

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Since the class of graphs is known to be Borel complete, this implies that the class of quandles is Borel complete.

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Dynamical quandles

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Then the operation $*$ on X given by

$$x * y = \begin{cases} y & \text{if } [x]_\tau \in \theta([y]_\tau) \\ \tau y & \text{if } [x]_\tau \notin \theta([y]_\tau) \end{cases}$$

makes $(X, *)$ a quandle, the *dynamical quandle derived from* (X, τ) *with respect to* θ.

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- $X = V \times 2$
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Then we define $Q(\Gamma)$ to be the dynamical quandle derived from (X, τ) with respect to θ.

Quandle definition

A quandle is a set with a binary operation $*$ such that

1. $\forall a \forall b \forall c[a * (b * c) = (a * b) * (a * c)]$
2. for all a and c there is a unique b such that $a * b = c$
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Equivalently, for every a the operation of left multiplication by a ($b \mapsto a * b$) is an automorphism with fixed point a.

Kei definition

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4. $\forall a \forall b[a * (a * b) = b]$

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Clearly if $\Gamma \cong \Gamma'$ then $Q(\Gamma) \cong Q(\Gamma')$.

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Interesting part: if there is an isomorphism $f : Q(\Gamma) \to Q(\Gamma')$, why must there be an isomorphism $\Gamma \to \Gamma'$?

Our isomorphism f need not arise from a graph isomorphism. Nevertheless, given f can we construct an isomorphism $\varphi : \Gamma \to \Gamma$?

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Then the "twinning" of (v, j) with $(v, 1 - j)$ is witnessed by the action of (u, i). So the action of $f(u, i)$ on $f(v, j)$ is nontrivial, and takes $f(v, j)$ it to *its* twin. So the first component of $f(v, j)$ is independent of $j \in \{0, 1\}$, and we take this to be $\varphi(v)$.

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These definitions of $\varphi(v)$ combine to produce a graph isomorphism from Γ to Γ'.

Open question

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Our map Q fails this badly, because graph homomorphisms need not preserve non-edges.